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THE RIEMANN APPROACH TO INTEGRATION: LOCAL GEOMETRIC THEORY

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The Riemann approach to integration

Local geometric theory

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This book presents a detailed and mostly elementary exposition of the generalized Riemann integrals discovered by Henstock, Kurzweil, and McShane. Along with the classical results, it contains some recent developments connected with lipeomorphic change of variables, higher-dimensional multipliers, and the divergence theorem for discontinuously differentiable vector fields.

Defining the Lebesgue integral in Euclidean spaces from McShane's point of view has a clear pedagogical advantage, since the initial stages of development are both conceptually and technically simpler. The McShane integral evolves directly from the basic ideas about integration taught in elementary calculus. The difficult transition from subdividing the domain to subdividing the range, intrinsic to the Lebesgue definition, is completely bypassed. The unintuitive Carathéodory concept of measurability is also made more palatable by means of locally fine partitions.

The mathematical significance of the generalized Riemann integrals emerges when the Henstock–Kurzweil approach is used to define the Denjoy–Perron integral. While there is little similarity between the classical definitions of Lebesgue, Denjoy, and Perron, the Lebesgue and Denjoy–Perron integrals are naturally connected via the McShane and Henstock–Kurzweil definitions. This is used to obtain a coordinate free multidimensional integral which provides an unrestricted Gauss–Green theorem for vector fields with large sets of singularities.

Although written as a monograph, the book can be used as a graduate text, and certain portions of it can be presented even to advanced undergraduate students with a working knowledge of limits, continuity, and differentiation on the real line.

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109 The Riemann approach to integration

To Lida

Preface

> If we think of the Lebesgue integral as God sent, then differentiable functions whose derivatives are not Lebesgue integrable may appear evil. The challenge is to resolve the conflict, particularly in higher dimensions.

This book presents a detailed and mostly elementary exposition of the generalized Riemann–Stieltjes integrals discovered by Henstock, Kurzweil, and McShane more than thirty years ago. Aside from the classical results, it contains some recent developments connected with lipeomorphic change of variables, higher-dimensional multipliers, and the divergence theorem for discontinuously differentiable vector fields.

Roughly speaking, the generalized Riemann integrals differ from the classical Riemann integral in that *uniformly fine* partitions of the integration domain are replaced by *locally fine* partitions. This idea is perhaps best explained by looking at the numerical evaluation of integrals by means of rectangular approximations. For instance, suppose we want to evaluate the integral

$$\int_{10^{-3}}^{10^3} t^{-1} \sin t^{-1} \, dt .$$

The behavior of the integrand in the intervals $[10^{-3}, 1]$ and $[1, 10^3]$ is very different: it oscillates rapidly in the first interval and decreases steadily to zero in the second. From this observation, it is easy to conclude that employing uniformly fine (e.g., equidistant) partitions would be wasteful. The most efficient evaluation of the integral is provided by a rectangular approximation based on a partition of $[10^{-3}, 10^3]$ that is *fine* in $[10^{-3}, 1]$ and *coarse* in $[1, 10^3]$. Such a partition is an example of a locally fine partition of the interval $[10^{-3}, 10^3]$. In general, the mesh of a locally fine partition varies from point to point. The full impact of this idea is well illustrated by Example 2.2.9 below.

The ingenious passage from uniformly fine to locally fine partitions has profound consequences: instead of the classical Riemann integral we obtain the Lebesgue and Denjoy–Perron integrals, depending upon whether the McShane or Henstock–Kurzweil approach is used.

While in the final analysis there is no substitute for the standard definition of the Lebesgue integral in an abstract measure space, defining the Lebesgue integral in Euclidean spaces from McShane's point of view has a clear pedagogical advantage: the initial stages of development are appreciably simpler, both conceptually and technically. The McShane integral evolves naturally from the initial ideas about integration we learn in the first courses of calculus. The traumatic transition from subdividing the domain to subdividing the range, intrinsic to the Lebesgue definition, is completely bypassed. The unintuitive Carathéodory concept of measurability is also made more palatable by means of locally fine partitions.

What has been said about the Lebesgue integral is true to an even larger degree when the Henstock–Kurzweil approach is used to define the Denjoy–Perron integral. While there is little similarity between the classical definitions of Lebesgue, Denjoy, and Perron, introducing the Lebsgue integral by McShane's definition provides a natural path to the more delicate integral of Henstock and Kurzweil. In my opinion, the successful multidimensional generalizations of the Denjoy–Perron integral obtained during the past decade are due mainly to the simplicity of the Henstock–Kurzweil definition.

Beyond advanced calculus, the prerequisites for understanding this book amount to little more than mathematical sophistication. My primary goal has been to make the material understandable to beginners whose background does not exceed the first year of graduate school. In fact, a large portion should be accessible to any undergraduate student who has a working knowledge of limits, continuity, and differentiation on the real line. On the other hand, it is only fair to admit that some familiarity with the Lebesgue integral will help the reader to appreciate the subject. While certain aspects of Perron and descriptive Denjoy integration enter implicitly into the exposition, the constructive Denjoy integral is never used.

Parts I and II of the book deal with one-dimensional and multidimensional integration, respectively. Although formally the one-dimensional McShane integral is a special case of the m-dimensional one, I believe that integration in higher dimensions is conceptually more complicated and often difficult to understand by those who are not properly initiated on the real line. This is particularly true for the conditionally convergent integrals of Henstock–Kurzweil type: here even the formal similarity breaks down when passing to dimensions greater than one. The unavoidable consequence of the two-level presentation is a certain amount of repetition, which I have strived to minimize. It is limited to those cases when repeating an argument from a slightly different point of view enhances understanding. Trivial repetitions are invariably left to the reader.

The book is organized into thirteen chapters. The first two are devoted to a completely elementary and self-contained development of the McShane integral on the real line, including the convergence theorems and integration by parts. These chapters, which elaborate my paper [39], are inspired by the original work of McShane (see [28] and [29]). They are fully accessible to more advanced undergraduate students; I have covered them successfully in several undergraduate courses.

In Chapter 3, locally fine partitions are used to define measures on the real line. This is conceptually more difficult but should present few problems to the readers who absorbed Chapters 1 and 2. All the basic properties of measures and measurable sets are carefully derived, and measurable functions are introduced. We reconcile the additivity of generalized length on nonoverlapping intervals with that of the induced measure on disjoint measurable sets.

The relationship between measure and the McShane integral is investigated in Chapter 4. We show that the measurability of a bounded set is equivalent to the integrability of its characteristic function, and that the value of the integral equals the measure of the set. Using sets of measure zero, we extend the definition of the integral to functions which have infinite values and are defined only almost everywhere. The main result is the Vitali–Carathéodory theorem, which is used to show that the McShane and Lebesgue integrals are equivalent.

Chapter 5 moves to the more demanding field of differentiation. It contains the covering theorems of Vitali and Besicovitch, the Radon–Nikodym theorem, and the Lebesgue decomposition of increasing functions.

The Henstock–Kurzweil integral is introduced in Chapter 6. We establish its main properties, including the relationship to the McShane integral. The central topic is the evolution of the fundamental theorem of calculus that leads to the multidimensional generalization presented in Chapter 11. Here the Stepanoff theorem plays an important role. We prove it in the real line only, but in such a way that the generalization to any dimension is obvious.

Chapters 7, 8, and 9, which are the beginning chapters of Part II, contain results concerning the multidimensional McShane integral. The divergence theorem is obtained independently of Fubini's theorem. Its proof is attuned to the spirit of generalized Riemann integration and points to further generalizations discussed in Chapters 11, 12, and 13.

In Chapter 10 we show that the McShane integral is invariant with respect to changes of coordinates that are more general than Lipschitz. This result is derived from the equivalence between the McShane and Lebesgue integrals, and does not have a direct relationship to McShane's definition.

Chapters 11, 12, and 13 deal with multidimensional generalizations of the Henstock–Kurzweil integral. They are motivated by the desire to recover the flux of any differentiable vector field by integrating its divergence, which may not be Lebesgue integrable. The topics are relatively new and will introduce the reader to current research in the area. For various reasons given in Chapter 13, it appears that the most useful generalization is the $\mathcal{BV}$-integral utilizing sets of finite perimeter. We sketch its definition and hint at some of its properties without going into details. Many interesting results concerning this integral have been obtained in recent years, but a definitive treatment awaits future development. The area remains a subject of vigorous research and, in my view, is not ready for presentation in book form. The interested reader is referred to papers [19], [20], [21], [24], [37], [36], [25], and [38].

Numerous exercises are scattered throughout the text, the majority of them containing easily provable results that form an integral part of the exposition. Thus a diligent effort should be made to work them out.

For completeness, proofs of several well-known theorems have been included. No originality is claimed: these proofs are simple adaptations of those found in standard texts such as [11], [43], [44], and [49].

I consciously avoided abstractions such as Henstock's division spaces, generalized limits, or functions with values in Banach spaces. The readers who wish to pursue a more abstract approach to Riemann integration may consult the books of Henstock ([14], [15], [16]), Kurzweil ([23]), and McShane ([28], [30]).

The present book contains no historical comments.

Davis, California W. F. P.

Acknowledgments

It is a pleasure to acknowledge the contributions of my teachers, colleagues, and friends towards the completion of this work.

I was introduced to nonabsolutely convergent integration by J. Mařík, and learned about the Riemann approach to it from the works of R. Henstock, J. Kurzweil, and the late E. J. McShane; many of my ideas grew out of personal contacts with McShane. The present book may not have been written, however, if it were not for the encouragement and moral support I received from P. S. Bullen and R. D. Mouldin during the time I worked on the divergence theorems for discontinuously differentiable vector fields.

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Several improvements are due to R. J. Battig and A. S. Jiang, who read various segments of the manuscript. E. J. Howard and J. W. Mortensen kindly agreed to undertake the tedious task of proofreading the final version and weeded out a multitude of misprints and errors; those which remain are the sole responsibility of the author. Throughout this writing V. H. DuBose and N. R. Staargaard selflessly provided on-line help with the intricacies of English grammar. In this regard I am also obliged to the Cambridge University Press, in particular to L. C. Gruendel and R. S. Wells, for their editorial help.

W. F. P.

Contents

Part I

One-dimensional integration

Chapter 1
Preliminaries

If E is a set and Π is a property of the elements of E, we denote by

$$\{x \in E : \Pi(x)\}$$

the set of all elements of E that have the property Π. A *countable set* is either finite or countably infinite. A *map* $f : X \to Y$ from a set X into a set Y is a set $f \subset X \times Y$ such that for each $x \in X$ there is a unique $y \in Y$ with $(x, y) \in f$. As usual, we write $y = f(x)$ instead of $(x, y) \in f$. We note that if $X = \emptyset$ (the empty set), then for any set Y there is a unique map $f : X \to Y$, namely the *empty map* $f = X \times Y = \emptyset$. An *enumeration* of a countable set C is a one-to-one map $n \mapsto c_n$ from a finite or infinite set $\{p, p+1, \ldots\}$ of integers onto C; in most cases $p = 1$, but this is not required.

The set of all real numbers is denoted by $\mathbf{R}$. An *interval* is a set

$$[a, b] = \{x \in \mathbf{R} : a \leq x \leq b\}$$

where $a, b \in \mathbf{R}$. We say that an interval $[a, b]$ is *degenerate* if $a \geq b$. A nondegenerate interval is called a *cell*. Thus an interval is a cell if and only if its interior is nonempty. The intersection of two intervals is again an interval; however, the intersection of two cells need not be a cell. A collection of intervals is called *nonoverlapping* if their interiors are disjoint. Note that a degenerate interval overlaps no interval (including itself). Aside from intervals, we shall consider the *segments*

$$(a, b) = \{x \in \mathbf{R} : a < x < b\},$$
$$[a, b) = \{x \in \mathbf{R} : a \leq x < b\},$$
$$(a, b] = \{x \in \mathbf{R} : a < x \leq b\}.$$