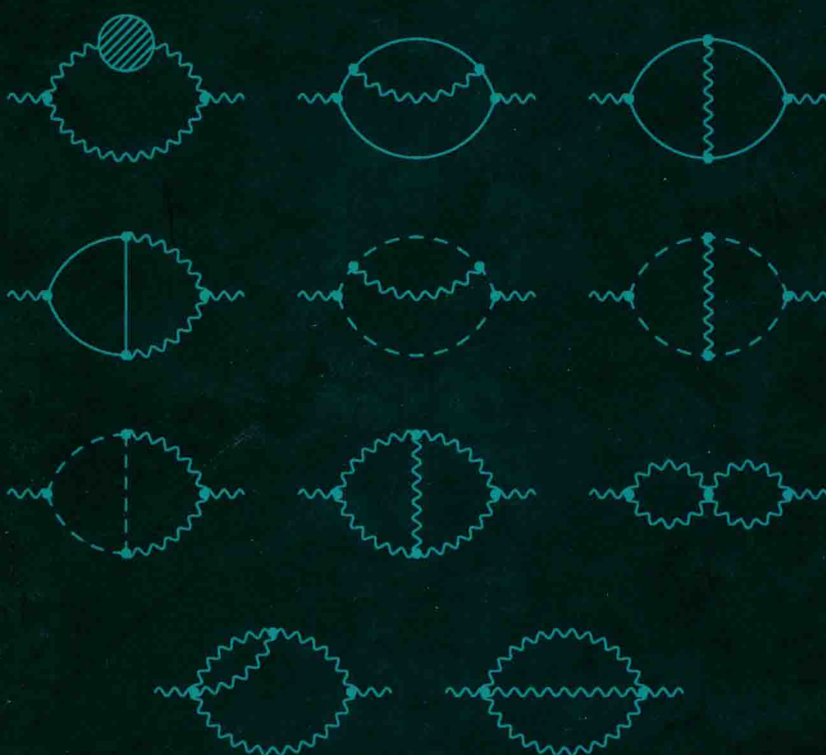
Lectures on QED and QCD

Practical Calculation and Renormalization of One- and Multi-Loop Feynman Diagrams



Andrey Grozin

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Preface

Precision of experimental data in many areas of elementary particle physics is quickly improving. For example, the anomalous magnetic moment of the muon is measured with a fantastically high precision. Results from B factories at SLAC and KEK for many quantities have low systematic errors and very high statistics. Of course, there are many more examples of such progress.

To compare high-precision experimental data with the theory, one has to obtain equally high-precision theoretical expressions for the measured quantities. Preparing physical programs for future colliders also requires high-precision theoretical calculations. In order to be able to search for a new physics, one has to understand standard processes (which can be a source of background) at a highly detailed level.

This means, in particular, calculation of higher radiative corrections. They are described by Feynman diagrams with one or several loops. Calculation of such diagrams is a very non-trivial task. It involves solving deep mathematical problems. Even when a suitable calculation algorithm has been constructed, this is not the end of the story. Often, many thousands of diagrams have to be calculated. This requires an unprecedented level of automation of theoretical research: generation and calculation of the diagrams have to be done systematically, by computer programs, without any interference of a human researcher. Some of the calculations of this kind are among the largest computer-algebraic calculations ever performed. This area of theoretical physics is progressing rapidly. A large number of physicists in many countries are involved in such activities. And this number is increasing. Many of today's students in the area of theoretical high energy physics will be involved in calculations of radiative corrections in the course of their careers.

Quantum field theory textbooks usually don't describe methods of calculation of multiloop Feynman diagrams. Most textbooks discuss quantization of fields (including gauge theories), obtain Feynman rules, and show a few simple examples of one-loop calculations. On the other hand, there is a huge amount of literature for experts in the area of multiloop calculations, usually in the form of original papers and specialized review articles. The purpose of this book is to close the gap between textbooks and the modern research literature. The reader should have a firm grasp of the basics of quantum field theory, including quantization of gauge fields (Faddeev–Popov ghosts, etc.) and Feynman rules. These topics can be found in any modern textbook, e.g., in [Peskin and Schroeder (1995)]. No previous experience in calculating Feynman diagrams with loops is required. Fundamental concepts and methods used for such calculations, as well as a large number of examples, are presented in this book in detail.

The main focus of the book is on quantum electrodynamics (QED) and quantum chromodynamics (QCD). In the area of QED, some extremely high-precision experimental data are available (anomalous magnetic moments, hydrogen atom, positronium). Correspondingly, some groundbreaking theoretical calculations have been done. In the area of QCD, very high precision comparisons of the theory and experiments are never possible, because we still don't know how to take non-perturbative phenomena into account quantitatively and in a model-free way (except by lattice Monte–Carlo simulations, whose accuracy is not very high but is increasing). However, the QCD coupling constant is much larger, and several terms of perturbative series are usually required to obtain the necessary (moderate) precision. Calculations in QED and QCD are usually very similar, but QCD ones are more lengthy — more diagrams, colour factors, etc. Therefore, a large fraction of the text is (technically) devoted to QED, but it should be considered also as a demonstration of methods which are used in QCD.

The first part of this book is based on lectures given to students preparing for the M. Sc. degree at Dubna International Advanced School on Theoretical Physics in 2005 and at Universität Karlsruhe. They were published as hep-ph/0508242. They were revised and extended for this book.

Practically all modern multiloop calculations are performed in the framework of dimensional regularization. It is discussed in Chap. 1, together with simple (but fundamentally important) one-loop examples. In Chaps. 2 and 3, one-loop corrections in QED and QCD are discussed. Here we use the $\overline{\text{MS}}$ renormalization scheme, which is most popular, especially in

QCD. Methods and results of calculation of two-loop corrections in QED and QCD are introduced in Chap. 4, also using $\overline{\text{MS}}$ scheme. Chap. 5 is devoted to the on-shell renormalization scheme, which is most often used in QED at low energies, but also for heavy-quark problems in QCD. Decoupling of heavy quarks is most fundamental in QCD; it is employed practically every time one does any work in QCD. It is presented in Chap. 6, where a simplified QED problem is considered in detail; it makes understanding the problem much easier. This is the first time decoupling in the $\overline{\text{MS}}$ scheme is considered in a textbook, with full calculations presented. Finally, Appendix A is a practical guide on calculating colour factors, which is a necessary (though simple) step in any QCD work. Here I follow an excellent book [Cvitanović (web-book)] available on the Web.

The second part is based on lectures given to Ph. D. students at the International School "Calculations at modern and future colliders", Dubna (2003), and at Universität Karlsruhe. They were published in Int. J. Mod. Phys. A **19** (2004) 473. They are (slightly) revised for this book. This second lecture course forms a natural sequel to the main one. It discusses some advanced methods of multiloop calculations; in cases when the same problem is discussed in both courses, it is solved by different methods. So, studying both of the courses gives a wider perspective and a better toolbox of methods. For a much more comprehensive presentation of modern methods of calculating Feynman integrals, the reader is addressed to a recent book [Smirnov (2006)].

Of course, there are a lot of things which are *not* discussed in this book. It only shows the most simple and fundamental examples. More complicated scattering processes (diagrams with more external legs) and radiative corrections in the electroweak theory (which often involve several kinds of particles with different masses) are not considered here. But the general approaches (dimensional regularization, $\overline{\text{MS}}$ renormalization, integration by parts...) remain the same. After reading this book, the reader should have no problems reading specialized literature about more advanced problems.

I am grateful to D.J. Broadhurst, K.G. Chetyrkin, A. Czarnecki, A.I. Davydychev, A.V. Smirnov, V.A. Smirnov for collaboration on various multiloop projects and numerous discussions, and to the organizers of the Dubna schools D.I. Kazakov, S.V. Mikhailov, A.A. Vladimirov for inviting me to give the lectures and for advices on the contents. A large part of the work on the book was done at the University of Karlsruhe, and was supported by DFG through SFB/TR 9; I am grateful to J.H. Kühn and M. Steinhauser for inviting me to Karlsruhe and fruitful discussions.

Andrey Grozin

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PART 1

QED and QCD

Chapter 1

One-loop diagrams

1.1 Divergences, regularization and renormalization

When interactions in a quantum field theory may be considered weak, we can use perturbation theory, starting from the theory of free fields in zeroth approximation. Contributions to perturbative series can be conveniently depicted as Feynman diagrams; corresponding analytical expressions can be reconstructed from the diagrams using Feynman rules. If a Feynman diagram contains a loop (or several loops), its expression contains an integral over the loop momentum (or several loop momenta). Such integrals often diverge at large loop momenta (*ultraviolet divergences*). For example, let's consider the scalar field theory with the $g\varphi^3$ interaction. The one-loop correction to the propagator (Fig. 1.1) is

$$g^2 \int \frac{d^4 k}{[m^2 - k^2 - i0]\,[m^2 - (p+k)^2 - i0]} \,. \tag{1.1}$$

At $k \to \infty$, the denominator behaves as k^4, and the integral diverges logarithmically.

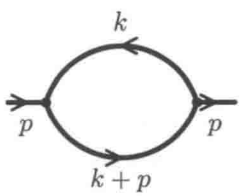

Fig. 1.1 One-loop propagator diagram

Therefore, first of all, we need to introduce a regularization — some modification of the theory which makes loop integrals convergent. One can sensibly manipulate and calculate regularized Feynman integrals. In the physical limit the original theory is restored. Then we should re-formulate the problem. The original perturbative expression (which contains divergent Feynman integrals, and hence is senseless) expresses a scattering amplitude via bare masses and charges which are present in the Lagrangian. But physical masses and charges don't coincide with these bare quantities, if we take radiative corrections into account. Therefore, we should fix some definition of physical masses and charges, and re-express our scattering amplitude via these quantities. This procedure is called renormalization. It is physically necessary, independently of the problem of divergences. If the theory makes sense, expressions for scattering amplitudes via physical masses and charges will remain finite when regularization is removed.

Of course, many different regularization methods can be invented. For example, a cutoff can be introduced into loop integrals by replacing propagators:

$$\frac{1}{m^2 - k^2 - i0} \to \frac{\theta\left(|k^2| < \Lambda^2\right)}{m^2 - k^2 - i0}.$$

The physical limit is $\Lambda \to \infty$. However, such a cutoff makes calculation of diagrams extremely difficult, because the integration region has a complicated shape. In addition to this, integrating by parts becomes very complicated because of boundary terms. Pauli–Villars regularization

$$\frac{1}{m^2 - k^2 - i0} \to \frac{1}{m^2 - k^2 - i0} - \frac{1}{M^2 - k^2 - i0}$$

(with the physical limit $M \to \infty$) is much better. However, it is not very good for gauge theories: gauge bosons in an (unbroken) gauge theory must be massless, and modifying their propagators by introducing massive terms breaks the gauge invariance.

In general, a good regularization method should preserve simple rules for manipulating loop integrals (like integration by parts), and also should preserve as much of symmetries of the theory as possible. Unbroken symmetries make calculations much simpler by restricting possible form of results. Sometimes, it is not possible to preserve *all* symmetries of a field theory when performing its regularization. In such a case, it may happen that renormalized results break some symmetry even in the limit of no regularization. This means that the quantum field theory has less symmetries than its classical Lagrangian suggests (an *anomaly*).

A popular regularization of gauge theories (after analytic continuation to Euclidean space-time) is to replace the continuous space-time by a cubic lattice with spacing a. The physical limit of this regularization is $a \to 0$. It can be done in an exactly gauge-invariant way invented by Wilson (matter fields live at lattice points, and gauge fields live on one-dimensional links). Within this approach, quantitative results can be obtained by Monte–Carlo simulation, without relying on perturbation theory. However, this regularization breaks Lorentz invariance (only a smaller symmetry group, that of a 4-dimensional cube, is preserved). This makes perturbative calculations much more difficult.

The most popular method used in multiloop calculations nowadays is *dimensional regularization*. Diagrams are calculated in d-dimensional space-time. The dimensionality d must appear in all formulas as a symbol, it is not enough to obtain separate results for a few integer values of d. The physical limit is $d \to 4$; therefore, d is often written as $4 - 2\varepsilon$. Divergences in intermediate perturbative formulas appear as $1/\varepsilon$ poles. After calculating a physical result in terms of physical parameters, we can take the limit $\varepsilon \to 0$ (this limit should exist in a sensible theory).

Dimensional regularization allows one to use simple algebraic rules for manipulating Feynman integrals. In particular, all integrations are over the whole infinite momentum space, and no surface terms appear during integration by parts. Dimensional regularization preserves Lorentz invariance (making it d-dimensional; when we take the limit $\varepsilon \to 0$ at the end of calculations, results automatically have a 4-dimensionally Lorentz-invariant form). In gauge theories, the d-dimensional Lagrangian is gauge invariant, so, the symmetry is preserved. Most other symmetries are preserved, too.

However, there are exceptions. As we shall see in Sect. 1.6, the Dirac matrix γ_5 cannot be generalized to d dimensions. Therefore, if we have a theory with massless fermions having chiral symmetry, this symmetry is not preserved in d dimensions, and in some cases it may be broken in final renormalized results at $\varepsilon \to 0$ (*axial anomaly*). Also, continuation to d dimensions changes dimensionalities of various quantities. Therefore, if the 4-dimensional massless theory was scale invariant, this symmetry (and a more general conformal symmetry) will be broken by regularization. This breaking can persist in renormalized results at $\varepsilon \to 0$ (*conformal anomaly*). Another important symmetry which is broken by dimensional regularization is supersymmetry. In supersymmetric theories, the numbers of bosonic and fermionic degrees of freedom coincide. However, these numbers depend on d in different ways, and supersymmetry is broken at $d \neq 4$.

Until now, we discussed ultraviolet divergences. In theories with massless particles (for example, gauge theories) some diagrams can also diverge at $k \to 0$ (*infrared divergences*)[1]. They cannot appear in results for meaningful physical quantities (we cannot detect arbitrarily soft photons, so, cross sections should be summed over final states with any number of such photons). In order to do intermediate manipulations, we have to regularize infrared divergences, too. This can be done by introducing a small photon mass; however, such a regularization breaks gauge invariance. Dimensional regularization regularizes infrared divergences as well as ultraviolet ones: both appear as $1/\varepsilon$ poles (in general, it is very difficult to trace which $1/\varepsilon$ poles are of ultraviolet origin and which are infrared).

1.2 Massive vacuum diagram

So, during these lectures, we are going to live in d-dimensional space–time: one time and $d-1$ space dimensions.

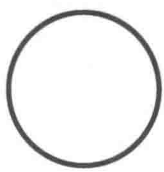

Fig. 1.2 One-loop massive vacuum diagram

Let's consider the simplest diagram shown in Fig. 1.2:

$$\int \frac{d^d k}{D^n} = i\pi^{d/2} m^{d-2n} V(n) \,, \quad D = m^2 - k^2 - i0 \,. \tag{1.2}$$

The power of m is evident from the dimensional counting, and our aim is to find the dimensionless function $V(n)$; we can put $m = 1$ to simplify the calculation. The poles in the complex k_0 plane are situated at

$$k_0 = \pm \left(\sqrt{\vec{k}^2 + 1} - i0 \right) \tag{1.3}$$

[1] If there is an on-shell massless particle with momentum p in the process, there can also be *collinear divergences* when the momentum k of a virtual particle is non-zero but parallel to p.