NONLINEAR PROBLEMS OF ENGINEERING

Edited by William F. Ames

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Foreword

The specific objective of the seminar which engendered this volume was to bring together a spectrum of engineering and applied mathematics scholars and to afford them and other attendants a forum for a review of certain nonlinear problems of engineering. The seminar, conducted by the Department of Mechanical Engineering, University of Delaware, was designed not primarily as an occasion for the presentation of research results but rather as one for exposition and discussion of nonlinear problems which occur in four areas: Mathematical Methods, Fluid Mechanics, Mechanics of Solids, and Transport Phenomena.

The sessions were held at the University of Delaware, Newark, Delaware, June 19–21, 1963. The attendants were welcomed on behalf of the University by President John A. Perkins, and Professor Jerzy Nowinski set the scientific tone of the meeting in his opening remarks. The seminar program was arranged and chaired by a faculty committee of the Department of Mechanical Engineering consisting of W. F. Ames, J. P. Hartnett, J. L. Nowinski, and B. S. Seidel.

The twenty lecturers were chosen on the basis of familiarity with their respective subjects, research contributions, and for ability as expositors. The seventy-five active participants, invited on a geographic basis, were especially chosen to ensure a stimulating, profitable discussion of current research, difficulties, and limitations of the fragmentary results in the nonlinear theory. Future lines of research and teaching were subjects of many formal and informal discussions.

Five of the lectures were not available for inclusion in the volume; they were:

Nonlinear Problems of Elastic Stability and Post Buckling Behavior, W. T. Koiter, Technical University, Delft, The Netherlands;

Some Orientable Material, J. L. Ericksen, Johns Hopkins University

Nonlinear Dispersive Waves, G. Whitham, California Institute of Technology

Some Recent Research in Nonlinear Elasticity, J. J. Stoker, New York University

On the Nonlinear Theory of Hydrodynamic Stability and Transition to Turbulence, C. C. Lin and D. J. Benney, Massachusetts Institute of Technology.

We extend thanks to all of the lecturers, participants, and guests. Special thanks are due to those who responded by travel from great distance. We wish also to thank the National Science Foundation for its support through grant GE-1314.

W. F. Ames　　J. L. Nowinski
J. P. Hartnett　　B. S. Seidel

Preface

Modern engineering, with refinements in instrumentation, advances in methods of computation, and its increasing venturesomeness, has made it evident that formulations of natural laws which neglect or suppress nonlinear terms often lead to inadequate or faulty results. Techniques are badly needed which incorporate the nonlinear terms in the equations.

Unfortunately for the engineer the assumption of linearity underlies a considerable domain of mathematics. Thus the mathematical tools available for application to the problems of the natural world are essentially linear. The great successes of the eighteenth and nineteenth centuries in constructing effective theories for physical phenomena were primarily due to a linear principle, that of superposition. However, these theories were only a first approximation to the true situation; that is, nature, with scant regard to the desires of mathematicians, seems to delight in formulating her mysteries in terms of nonlinear systems of equations.

In the past the term "nonlinear mechanics" has been applied to a series of investigations in the field of nonlinear ordinary differential equations which have had their origin, for the most part, in applications to physical problems. The literature of this area is now quite extensive and the research activity both by engineers and mathematicians, is also extensive. This research has been largely motivated by an engineering need for more precise results which are only available through a nonlinear theory.

The need for techniques in nonlinear continuum mechanics is acute. The literature is very sparse in these areas primarily because the mathematical model is inevitably a nonlinear partial differential or integral equation or combination. Properties of some of these systems as to the existence and uniqueness of and bounds on solutions have been investigated. This work continues at an accelerated pace. Our knowledge is, however, still very fragmented as can be seen from our infinitesimal knowledge of the important Navier–Stokes equations.

General principles, especially in the area of nonlinear partial differential equations, are still few in number. Methods of solution of the associated initial value and boundary value problems are most often, *ad hoc*, approximate or numerical with all of the uncertainties that such methods create. The loss of superposition is a severe one.

The history of nonlinear methods is largely unwritten. It is hoped that the contributions to that history, presented in this volume, will be valuable reference material in the present and in the future.

W. F. Ames

July 1964

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New Methods in Nonlinear Mechanics

RICHARD BELLMAN
The RAND Corporation
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I.

Introduction

Mathematics assumes many roles in the culture of our society, and even within science itself, simultaneously both handmaiden and queen. Among these functions are the tasks of providing conceptual and analytical frameworks for scientific theories and of furnishing algorithms for obtaining numerical answers to numerical questions.

It is not sufficiently emphasized, particularly in the undergraduate and graduate schools, that these various functions are intimately related. We have seen in the past, notably in connection with quantum mechanics and relativity theory, that numerical calculations combined with experiment can overthrow elegant classical theories and point the way to the development of still more elegant modern theories. In a similar fashion, contemporary needs for greater precision and greater understanding point the way to the development of new mathematical theories which will yield the desired information. In particular, it is to be expected that modern mathematical theories will take account of the existence of new devices available for numerical work, such as digital and analog computers.

In what follows, we wish to present some of the fundamental ideas of two new mathematical theories which have already provided powerful computational approaches [1–6]. These theories, *invariant imbedding* and *dynamic programming*, are based upon new conceptual approaches to classical analysis and mathematical physics.

II.

The Comparative Approach

A basic device in the study of phenomena in all fields is the use of families of processes to illuminate the behavior of an individual process. Thus, we see Comparative Anatomy, Comparative Philology, and, as a most important example, the theory of evolution in biology.

In the application of this mode of thought there are two crucial steps. The first is the recognition of families of processes which can be used for this imbedding, and the second is the derivation of useful relations connecting various members of the engulfing family.

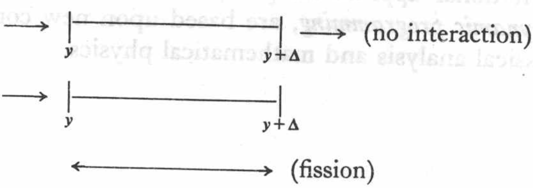
It is essential to keep in mind that a particular process can be imbedded in many different ways, just as a finite-dimensional object can be regarded as a cross-section of infinitely many higher dimensional constructs.

We shall illustrate the advantages and disadvantages of different types of imbedding by means of an idealized neutron transport process.

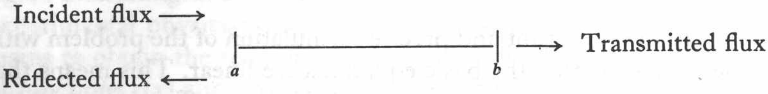
III.

Classical Approach

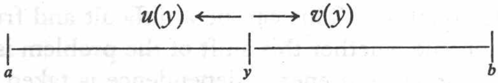
Let us consider an idealized version of a neutron transport process in which neutrons can move only to the left or right. This corresponds then to motion in a rod. We suppose that when a neutron enters the interval $[y, y+\Delta]$ from either direction (where Δ is an infinitesimal), there is a probability $(1-p(y)\Delta)+o(\Delta)$ of no interaction and a probability $p(y)\Delta+o(\Delta)$ of "fission," by which we mean that the original particle disappears and is replaced by two like particles, one moving to the right and one to the left.



Assuming that there is a constant flux of particles per unit time incident at the left end point of a finite rod $[a, b]$, we wish to determine the expected reflected flux and the expected transmitted flux, and to ascertain whether or not there is a "critical length."

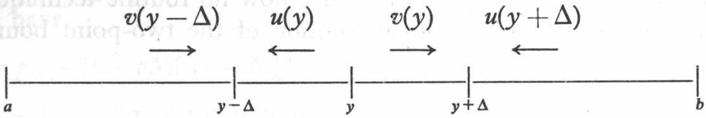


To treat these particular problems, we consider the family of problems where we wish to determine the expected right-hand and left-hand fluxes at a particular point y in the rod.



The value $u(a)$ is the desired reflected flux and $v(b)$ is the transmitted flux. This is certainly a meaningful imbedding.

It remains to obtain relations connecting the various members of the family of problems. Since there will be many such relations in general, we choose those that are particularly suited to our analytic and computational abilities. These restrictions dictate the use of differential or difference equations. Let us see then if we can relate $u(y)$ and $v(y)$ to $u(y \pm \Delta)$ and $v(y \pm \Delta)$.



On the basis of the assumptions we have made concerning local interactions, we obtain the following relations:

$$\begin{aligned} u(y) &= (1 - p(y)\Delta)u(y + \Delta) + p(y)\Delta(u(y) + v(y)) \\ v(y) &= (1 - p(y - \Delta)\Delta)v(y - \Delta) + p(y)\Delta(u(y) + v(y)) \end{aligned} \tag{3.1}$$

To obtain differential equations, we let $\Delta \to 0$. The resulting system of differential equations is

$$\begin{aligned} u'(y) &= -p(y)v(y) \\ v'(y) &= p(y)u(y) \end{aligned} \tag{3.2}$$

with the two-point boundary-value condition

$$v(a) = 1, \qquad u(b) = 0 \tag{3.3}$$

Criticality will exist if there is a value of b for which $u(y)$ and $v(y)$ become infinite for all y inside $[a, b]$.

IV.

Discussion

This is a very elegant and precise formulation of the problem with the prime advantage that the basic equations are linear. This means that we can use superposition techniques and, if the coefficients are constant, transform techniques. Even in this last case, however, there are major difficulties to overcome. The numerical solution of linear differential equations can be reduced in a routine way to the numerical solution of linear algebraic equations. This can be a difficult and frustrating affair, and it is questionable whether this shift of the problem is a reduction in many cases. If angular and energy dependence is taken account of, we obtain either a very large-dimensional system of ordinary differential equations, linear partial differential equations, or linear integro-differential equations.

In those problems where the Laplace transform of the solution can be obtained in explicit analytic form, there is still the major obstacle of obtaining a numerical inversion of the Laplace transform; see [7] and [8] for a discussion of these matters.

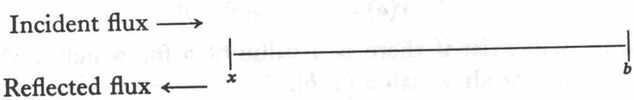
If we allow interactions between particles, the classical equations of transport become nonlinear. There are now no routine techniques for either analytic or computational solution of the two-point boundary-value problem.

V.

The New Approach of Invariant Imbedding

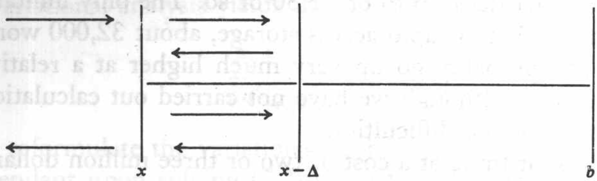
Let us, in view of what we have said above, consider a new method of imbedding which eliminates two-point boundary value aspects and provides an analytic formulation solely in terms of initial values.

To answer the problem as posed above of determining the reflected and transmitted fluxes associated with a rod of prescribed length $[a, b]$, we consider now the general problem of determining the reflected flux from a rod of arbitrary length $[x, b]$, $-\infty < x < b$.



The reflected flux depends upon x, and we emphasize this fact by the notation $r(x)$. Similarly the transmitted flux is denoted by $t(x)$. This is once again a meaningful imbedding and, indeed, the imbedding used by the experimental physicist in the study of these processes.

It remains to obtain the relations connecting various members of the family of functions $r(x)$ and $t(x)$. Consider the diagram below.



The flux $r(x)$ may be considered to be obtained in the following way:

(a) reflection on $[x, x - \Delta]$
(b) reflection from $[x - \Delta, b]$
(c) reflection of the flux $r(x - \Delta)$ in $[x - \Delta, x]$
(d) reflection of this last reflected flux from $[x - \Delta, b]$ (5.1)

All other reflections yield contributions of order Δ^2 and thus can be neglected. Adding up the contributions of the interactions mentioned above, we have

$$r(x) = p\Delta + (1 - p\Delta)[r(x - \Delta)(1 - p\Delta) + r^2(x - \Delta)p\Delta] + O(\Delta^2) \quad (5.2)$$

Letting $\Delta \to 0$, we obtain the nonlinear differential equation

$$r'(x) = p(x)(1 + r^2(x)) \quad (5.3)$$

with the initial condition $r(0) = 0$ [9–11].

VI.

Discussion

In the particular case discussed above, we can obtain an explicit analytic solution. This is of little importance. In the more general case where we take account of angular and energy dependence, the one-dimensional Riccati equation is replaced by the matrix Riccati equation

$$R'(x) = A(x) + B(x)R + RB(x) + RC(x)R, \qquad R(0) = 0 \quad (6.1)$$

The solution of this equation can by means of a known simple transformation be reduced to the solution of linear differential equations. This, however, is not feasible if the dimension of R is large, since the ultimate determination of R requires an inversion of a matrix of the dimension of R.

The computational solution of (6.1) is a routine matter using a digital computer for matrices up to order 50 or so. The only limitation at the present time is that of rapid access storage, about 32,000 words. Using tapes, we can probably go up very much higher at a relatively small increase in time. Although we have not carried out calculations of this nature, we foresee no difficulties.

At the present time, at a cost of two or three million dollars, current computers can be "souped up" to have rapid access storages of 10^6. In the foreseeable future, ten years or so, we can expect rapid access storages of 10^7 to 10^8. With these capabilities, we will be able to handle matrices of dimension 500 or 1000 and thus be far ahead of what the experimental physicist is supplying in the way of data.

Let us emphasize the point that there is great merit to using both imbeddings simultaneously, since each possesses certain useful and desirable features, both analytically and computationally. Furthermore, it is natural to suppose that there exist many other types of imbedding with other useful features. An important byproduct of this new approach is the realization that there is nothing sacrosanct about any particular mathematical formulation of any physical process. Each formulation must face the test of analytic and computational feasibility.

VII.

Variational Processes

Let us now apply the foregoing concepts to the study of variational processes. Consider the problem of minimizing the functional

$$J(u) = \int_a^b g(u, u', t)\, dt \tag{7.1}$$

over all functions $u(t)$ such that $u(a) = c$. The usual approach, modeled after the finite-dimensional variational procedure, leads to the Euler equation

$$\frac{\partial g}{\partial u} - \frac{d}{dt}\left(\frac{\partial g}{\partial u'}\right) = 0 \tag{7.2}$$

a second order nonlinear differential equation subject to various types of two-point boundary-value conditions dependent upon the initial assumptions. If, for example, we fix $u(a)$ to be c, but leave $u(b)$ variable, then the variational analysis yields the second condition

$$\left.\frac{\partial g}{\partial u'}\right|_{t=b} = 0 \tag{7.3}$$

Can we reformulate the variational problem so as to avoid the difficulties attendant upon solving (7.2) subject to (7.3) and $u(a) = c$, and the perhaps even more serious matter that (7.2) is only a sufficient condition?

As we shall see, we can, using the theory of dynamic programming, a theory which enables us to extend variational techniques to handle stochastic and adaptive processes as well.

VIII.

Geodesics

To illustrate the principal ideas unencumbered by analytic details, let us consider the problem of determining a path in phase space of minimum time. Let p_1 and p_2 be the end points of the path (Fig. 1).

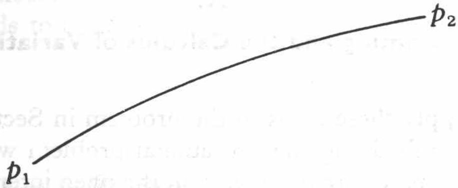


FIG. 1.

To treat this problem using imbedding concepts, we consider the more general problem of determining the minimal time to go from an