Analytical Techniques in the Theory of Guided Waves

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Roger F. Harrington, Editor

Preface

There has been considerable progress recently toward the development of analytical techniques for attacking boundary-value problems in guided-wave theory. This book is an attempt to present a unified account of a number of these techniques. It is intended to serve as a graduate-level text, and it is hoped that the book will also be found useful by researchers in the areas of electromagnetics and acoustics. The emphasis is on elaborating the principles of various mathematical techniques rather than on solving a large number of specific problems. Thus the same geometrical configuration is frequently chosen to serve as a typical example for the application of more than one analytical technique that may be employed to attack a problem. This permits convenient comparison of both the methods and the formats of solutions derived by these techniques. A large number of exercises have been included, accompanied by hints for solving them. These exercises deal with physical phenomena associated with the study of waveguide discontinuity, radiation and diffraction, and array problems and reference is made to the original papers that discuss these problems.

The book begins with the presentations of preliminary and background material in Chapter 1. Although some of this material is available in other texts, its inclusion is mainly for convenience of later reference. However, it is believed that Section 1-3, on the edge condition, and Section 1-4, on useful asymptotic formulas, contain materials that cannot be conveniently located in other sources.

Chapter 2 deals with the mode-matching technique, one of the most commonly used methods for formulating boundary-value problems in guided-wave theory. It illustrates the application of the direct-inversion and the residue-calculus methods for the exact solution of a class of problems involving waveguides and periodic structures. Chapters 2 and 3, both of which deal with classes of problems that possess exact solutions, serve as important background material for discussion of more advanced techniques in Chapters 4 and 5—techniques that are concerned with the derivation of semirigorous solutions to a much wider class of problems, which do not lend themselves to exact solution.

Chapter 3 presents a rather comprehensive discussion of the Wiener–Hopf technique based on the application of Fourier transforms and the theory of analytic continuation in complex variable theory. The Wiener–Hopf technique is typically discussed only very briefly, or in passing, in many texts on electromagnetic theory, although the technique provides a powerful tool for solving a host of boundary-value problems that conform to a special type of geometr) The style of presentation of this material is somewhat different from that c Noble's text on Wiener–Hopf technique, although our Chapter 3 makes fre quent use of the material appearing in that excellent book. Chapter 3 als includes some new forms of factorization formulas which enhance the use fulness of the Wiener–Hopf method. An additional feature is the discussio that establishes the connection between the Wiener–Hopf and the mode matching techniques.

Chapters 4 and 5 are concerned with the generalization of the mode-match ing and Wiener–Hopf techniques, with a view to significantly broadenin their ranges of applications. The modified residue-calculus method discusse in Chapter 4 was developed only very recently but has found man applications to problems related to open and closed region waveguide di: continuity problems, phased arrays, and other periodic structures, particularl when applied in conjunction with the generalized scattering-matrix technique also discussed in this chapter. Another contribution of Chapter 4 is a de scription of the generalization of the mode-matching technique to open-regio problems.

Finally, the Wiener–Hopf technique is generalized in Chapter 5 so as to b useful for a wider class of geometries. The generalization includes a cas where the filling medium in the waveguide is inhomogeneous, a situation tha has not been previously discussed elsewhere using the Wiener–Hopf technique

Two other methods, the variational and quasi-static techniques, hav been omitted from the list of topics because extensive discussions are readil available in a number of texts and reference books.

During the course of preparation of this book, the authors received er couragement and helpful criticism from many colleagues and friends at th University of Illinois, Professor G. A. Deschamps, Professor Y. T. Lo, an Mr. T. S. Li in particular. The book draws heavily upon the research publ: cations and dissertations of former research students at the University c Illinois: Drs. J. R. Pace, D. S. Karjala, C. P. Bates, G. F. VanBlaricum, Jr T. Itoh, and others. To them, the authors are deeply indebted. The expe secretarial help of Mrs. Lilian Beck, Mrs. Sharon Gocking, Mrs. Avi Opheim, and Mrs. Angie Johnson was much appreciated during the severa stages of the preparation of the manuscript as the text evolved over a perio of two years. Finally, much of the research work included in the text wa sponsored by the Air Force Cambridge Research Laboratories under th monitorship of Mr. F. Zucker and Dr. R. A. Shore. The authors take thi opportunity to express their thanks for the financial assistance received fror the AFCRL through contract support.

Urbana, Illinois R. M. and S. W. L.

Basic Conventions and Notations

1. MKS units and $e^{-i\omega t}$ time variation are used throughout (Section 1-2).

2. $\psi(x, z)$ usually may be identified by $H_y(x, z)$, and $\phi(x, z)$ by $E_y(x, z)$ (Section 1-5).

3. $\gamma = (\alpha^2 - k^2)^{1/2} = -i(k^2 - \alpha^2)^{1/2}$, where $\alpha = \sigma + i\tau$ and $k = k_1 + ik_2$ (Section 1-6).

4. The Fourier transform pair is (Section 3-2)

$$\Psi(x, \alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x, z) e^{i\alpha z} \, dz$$

$$\psi(x, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x, \alpha) e^{-i\alpha z} \, d\alpha$$

5. C = Euler's constant = 0.57721... (Section 1-4).

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Contents

CHAPTER 1

Preliminaries

1-1 Introduction

The purpose of this book is to give a unified account of a number of analytical methods that are found useful for solving a large class of open and closed waveguide problems. As a preparatory step we will present in this chapter some basic concepts and formulas to which subsequent discussions will frequently refer. The material included in Sections 1-2, 1-5, 1-6, and 1-7 can be found in several graduate-level textbooks and therefore serves only as a brief reminder. However, some of the discussion on edge condition, presented in a self-contained manner in Sections 1-3 and 1-4, may not be as readily accessible from other sources. The advanced reader may choose to bypass this chapter entirely and return to it when a particular reference is necessary.

1-2 Maxwell's Equations

The behavior of the electromagnetic fields in a continuous medium, isotropic or anisotropic, homogeneous or inhomogeneous, is governed by Maxwell's equations. In the MKS system of units, the differential forms of Maxwell's equations are

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2.1}$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} \tag{2.2}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2.3}$$

$$\nabla \cdot \boldsymbol{D} = \tilde{\rho} \tag{2.4}$$

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \tilde{\rho}}{\partial t} \tag{2.5}$$

with the various quantities defined as

$\boldsymbol{E}$: *electric field* (in volts per meter)
$\boldsymbol{H}$: *magnetic field* (in amperes per meter)
$\boldsymbol{D}$: *electric flux density* (in coulombs per square meter)
$\boldsymbol{B}$: *magnetic flux density* (in webers per square meter)

J: *electric current density* (in amperes per square meter)
$\tilde{\rho}$: *electric charge density* (in coulombs per cubic meter)

It is to be noted that these five equations, (2.1) through (2.5), are not a independent. For example, with an appropriate initial condition we can ob tain (2.3) by taking the divergence of (2.1). In a similar manner (2.4) may b derived from (2.2), in conjunction with (2.5) and appropriate initial condition:

Throughout this book we will be concerned with *time-harmonic electro magnetic fields* only, and we will assume that all field quantities have a tim variation given by $\exp(-i\omega t)$, where ω is the angular frequency in radians.

Under this assumption the time derivatives in Maxwell's equations may b replaced by the factor $-i\omega$, and the common factor $\exp(-i\omega t)$ may be droppe from these equations. Equations (2.1) through (2.5) then become

$$\nabla \times \mathbf{E} = i\omega\mathbf{B} \quad (2.6$$

$$\nabla \times \mathbf{H} = -i\omega\mathbf{D} + \mathbf{J} \quad (2.7$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.8$$

$$\nabla \cdot \mathbf{D} = \rho \quad (2.9$$

$$\nabla \cdot \mathbf{J} = i\omega\rho \quad (2.10$$

Note that we have used boldface roman letters for the vectors that are comple functions of space coordinates only.

The set of Maxwell's equations given in (2.6) through (2.10) is not sufficien to determine the electromagnetic fields produced by given current and charg densities. The additional equations necessary for this purpose are supplie by relations between the fields (**E**, **H**), the flux densities (**D**, **B**), and the curren density **J**, and are determined by the properties of the material medium in volved.† These relations are generally known as the *constitutive relations* o the medium. The media are usually grouped into two categories: isotropi and anisotropic media.

Isotropic media. An isotropic medium is generally regarded as one in whic the physical properties in the neighborhood of an interior point are the sam in all directions. For most isotropic media, the constitutive relations are

$$\mathbf{D} = \epsilon\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}, \qquad \mathbf{J} = \sigma\mathbf{E} \quad (2.11$$

If ϵ, μ, and σ are not functions of position, the medium is said to be homo geneous; otherwise, it is called an inhomogeneous medium. The free space, o vacuum, is an isotropic medium in which

$$\epsilon = \epsilon_0 \approx \frac{1}{36\pi \times 10^9} \text{ farads/meter} \quad (2.12a$$

† These relations are also dependent on the frames of reference of the mediun and the observer if there is relative motion between these two frames.

$$\mu = \mu_0 = 4\pi \times 10^{-7} \text{ henrys/meter} \tag{2.12b}$$

$$\sigma = 0 \tag{2.12c}$$

For other isotropic media, it is convenient to introduce the dimensionless ratios

$$\epsilon_r = \frac{\epsilon}{\epsilon_0}, \qquad \mu_r = \frac{\mu}{\mu_0} \tag{2.13}$$

which are generally labeled the relative dielectric constant and relative permeability, respectively.

Anisotropic media. In an anisotropic medium, the physical properties in the neighborhood of a point may be different for different directions. Typical examples are crystals, magnetized ferrites, and ionized media with externally applied static magnetic fields. Their constitutive relations can be generally represented by

$$\mathbf{D} = \boldsymbol{\epsilon} \cdot \mathbf{E}, \qquad \mathbf{B} = \boldsymbol{\mu} \cdot \mathbf{H}, \qquad \mathbf{J} = \boldsymbol{\sigma} \cdot \mathbf{E} \tag{2.14}$$

where $\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$, and $\boldsymbol{\sigma}$ are tensors of rank two, alternatively termed dyadics. We will not deal with anisotropic media except in certain exercises, at which time we will present the explicit forms of $\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$, and $\boldsymbol{\sigma}$ for some special anisotropic media.

In dealing with guided-wave problems, we often face a situation in which the physical properties of the medium change abruptly across one or several surfaces. The behavior of the fields in the presence of such discontinuities is governed by certain boundary conditions to be satisfied at the surfaces of the discontinuities. These conditions may be derived by an application of Maxwell's equations to infinitesimally small regions containing these surfaces. Some explicit forms of boundary conditions are as follows:

1. *At a material boundary* (discontinuous ϵ and μ; refer to Figure 1-1). If the media in regions 1 and 2 have finite conductivities, the tangential electric and magnetic components are continuous across the boundary. That is,

$$\mathbf{n} \times (\mathbf{E}^{(2)} - \mathbf{E}^{(1)}) = 0 \tag{2.15a}$$

$$\mathbf{n} \times (\mathbf{H}^{(2)} - \mathbf{H}^{(1)}) = 0 \tag{2.15b}$$

where $\mathbf{n}$ is the unit outward normal viewed from region 1.

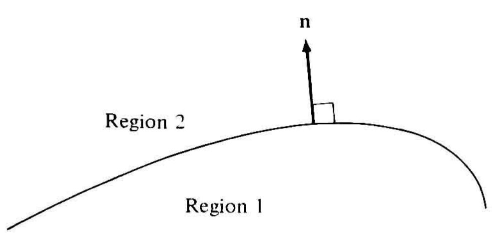

FIGURE 1-1 Boundary surface.

2. *At a perfectly conducting surface.* Let the medium in region 1 in Figu 1-1 be a perfect conductor (with an infinitely large conductivity σ). Then t boundary conditions are

$$\mathbf{n} \times \mathbf{E}^{(2)} = 0 \qquad (2.16$$

$$\mathbf{n} \times (\mathbf{H}^{(2)} - \mathbf{H}^{(1)}) = \mathbf{J}_s \qquad (2.16$$

where $\mathbf{J}_s$ is the surface current density.

1-3 Radiation Condition and Edge Condition

In certain situations in which the region of interest either involves boundar at infinity or contains geometrical singularities, it is possible to derive seve mathematically acceptable solutions of Maxwell's equations, only one which is consistent with anticipated physical phenomenon. Therefore, in the situations it becomes necessary to apply certain additional physical constrai to ensure the uniqueness of the solutions.

In an unbounded space with all sources contained in a finite region, t additional constraint that governs the behavior of the fields at infinity is stat in terms of the *radiation condition*, which may be applied in one of two wa If the medium in the space is lossy, we require that *the fields vanish at infini* If the medium is lossless and isotropic, the behavior of the fields at infinity governed by the *Sommerfeld radiation condition*, which may be stated follows. The field at a large distance r from the source has a phase progressi outward and has an amplitude that decreases at least as rapidly as r^{-1}. Mo precisely, any transverse components ψ of the field (with respect to the direction) must satisfy the condition

$$\lim_{r\to\infty} r\left(\frac{\partial\psi}{\partial r} - ik\psi\right) = 0 \qquad (3$$

where $k = \omega\sqrt{\mu\epsilon}$ is the propagation constant of the medium.

Yet another situation, where the solution of Maxwell's equations may not unique, arises when the configuration of the problem contains geometric singularities, such as sharp edges. The additional physical condition need here, known as the *edge condition*, is supplied by the requirement that t electrical and magnetic energy stored in any finite neighborhood of the ed must be finite; that is,

$$\int_V (\epsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2)\, dv \to 0 \qquad (3$$

as the volume V contracts to the neighborhood of the edge. For a smoo edge, which may be regarded as locally straight, the differential volume

(3.2) is $dv = \rho\, d\rho\, d\phi\, dz$, where (ρ, ϕ, z) is the locally cylindrical coordinate of the edge. Then from (3.2) one may deduce that in the neighborhood of the edge, *none of the field components of* (**E**, **H**) *should grow more rapidly than* $\rho^{-1+\tau}$ *with* $\tau > 0$ *as* $\rho \to 0$. Strictly speaking, it is not necessary to know a priori the exact value of τ but only its lower bound, which is greater than zero, in order to derive a unique solution to Maxwell's equations. In many instances, however, it is convenient to have a prior knowledge of τ. We will now illustrate how the characteristic value τ can be calculated from Maxwell's equations and the knowledge of the edge configuration. The method we follow is based on a study by Meixner.† For problems encountered in this book, it will be sufficient to consider, for the purpose of determining τ, a two-dimensional perfectly conducting wedge as shown in Figure 1-2. The three media surrounding the edge in regions 1, 2, and 3 are characterized by (μ_1, ϵ_1), (μ_2, ϵ_2), and

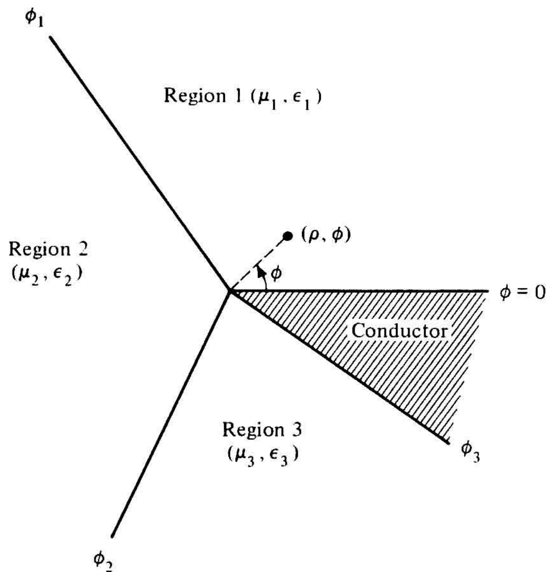

FIGURE 1-2 Perfectly conducting wedge surrounded by three different isotropic media.

(μ_3, ϵ_3), respectively. The angles, ϕ_1, ϕ_2, and ϕ_3, are all defined between 0 and 2π. Maxwell's equations (2.6) and (2.7), together with the constitutive relations (2.11), may be written in the cylindrical coordinate system (ρ, ϕ, z) as follows:

† J. Meixner, "The Behavior of Electromagnetic Fields at Edges," Inst. Math. Sci. Res. Rept. EM-72, New York University, New York, N.Y., Dec. 1954.

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} = i\omega\mu H_\rho$$

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = i\omega\mu H_\phi \qquad (3$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\phi) - \frac{1}{\rho}\frac{\partial E_\rho}{\partial \phi} = i\omega\mu H_z$$

$$\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} = -i\omega\epsilon E_\rho$$

$$\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = -i\omega\epsilon E_\phi \qquad (3$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi) - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi} = -i\omega\epsilon E_z$$

Since the field behavior near the edge $\rho = 0$ is of interest, we may expa each of the field components in different angular regions as a power series ρ. Recall from the edge condition that in the neighborhood of the edge components of the field can grow more rapidly than $\rho^{-1+\tau}$, with $\tau > 0$. W this in mind we may write

$$E_\rho = \rho^{-1+\tau}[a_0^{(j)} + a_1^{(j)}\rho + a_2^{(j)}\rho^2 + \cdots]$$

$$E_\phi = \rho^{-1+\tau}[b_0^{(j)} + b_1^{(j)}\rho + b_2^{(j)}\rho^2 + \cdots] \qquad (3$$

$$E_z = \rho^{-1+\tau}[c_0^{(j)} + c_1^{(j)}\rho + c_2^{(j)}\rho^2 + \cdots]$$

$$H_\rho = p^{-1+\tau}[A_0^{(j)} + A_1^{(j)}\rho + A_2^{(j)}\rho^2 + \cdots]$$

$$H_\phi = \rho^{-1+\tau}[B_0^{(j)} + B_1^{(j)}\rho + B_2^{(j)}\rho^2 + \cdots] \qquad (3$$

$$H_z = \rho^{-1+\tau}[C_0^{(j)} + C_1^{(j)}\rho + C_2^{(j)}\rho^2 + \cdots]$$

where $j = 1$, 2, and 3, corresponding to fields in the three regions of Figı 1-2. The coefficients in (3.5) and (3.6) are functions of ϕ and z only. Soı of the relations between these coefficients can be determined by inserti (3.5) and (3.6) into Maxwell's equations in (3.3) and (3.4) and comparing t coefficients of equal powers of ρ. When this is done, the following relatic are obtained:

$$c_0^{(j)}(\tau - 1) = 0 \qquad (3.$$

$$-i\omega\epsilon_j a_0^{(j)} = \frac{\partial C_1^{(j)}}{\partial \phi} - \frac{\partial B_0^{(j)}}{\partial z} \qquad (3.$$

$$-i\omega\epsilon_j b_0^{(j)} = \frac{\partial A_0^{(j)}}{\partial z} - \tau C_1^{(j)} \qquad (3.$$

$$0 = \tau B_0^{(j)} - \frac{\partial A_0^{(j)}}{\partial \phi} \qquad (3.$$