LINEAR ALGEBRA

THEOREMS AND APPLICATIONS

Edited by **Hassan Abid Yasser**

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Preface

The core of linear algebra is essential to every mathematician, and we not only treat this core, but add material that is essential to mathematicians in specific fields. This book is for advanced researchers. We presume you are already familiar with elementary linear algebra and that you know how to multiply matrices, solve linear systems, etc. We do not treat elementary material here, though we occasionally return to elementary material from a more advanced standpoint to show you what it really means. We have written a book that we hope will be broadly useful. In a few places we have succumbed to temptation and included material that is not quite so well known, but which, in our opinion, should be. We hope that you will be enlightened not only by the specific material in the book but also by its style of argument. We also hope this book will serve as a valuable reference throughout your mathematical career.

Chapter 1 reviews the metric Hermitian 3-algebra, which has been playing important roles recently in sting theory. It is classified by using a correspondence to a class of the super Lie algebra. It also reviews the Lie and Hermitian 3-algebra models of M-theory. Chapter 2 deals with algebraic analysis of Appell polynomials. It presents the determinantal approaches of Appell polynomials and the related topics, where many classical and non-classical examples are presented. Chapter 3 reviews a universal relation between combinatorics and the matrix model, and discusses its relation to the gauge theory. Chapter 4 covers the nonnegative matrices that have been a source of interesting and challenging mathematical problems. They arise in many applications such as: communications systems, biological systems, economics, ecology, computer sciences, machine learning, and many other engineering systems. Chapter 5 presents the central theory behind realization-based system identification and connects the theory to many tools in linear algebra, including the QR-decomposition, the singular value decomposition, and linear least-squares problems. Chapter 6 presents a novel iterative-recursive algorithm for computing GI for block matrices in the context of wireless MIMO communication systems within RFC. Chapter 7 deals with the development of the theory of operator means. It setups basic notations and states some background about operator monotone functions which play important roles in the theory of operator means. Chapter 8 studies a general formulation of Jensen's operator inequality for a continuous field of self-adjoint operators and a field of positive linear mappings. The aim of chapter 9 is to present a system of linear equation and inequalities in max-algebra. Max-algebra is an analogue of linear algebra developed on a pair of operations extended to matrices and vectors. Chapter 10 covers an efficient algorithm for the coarse to fine scale transition in multi-flexible-body systems with application to biomolecular systems that are modeled as articulated bodies and undergo discontinuous changes in the model definition. Finally, chapter 11 studies the structure of matrices defined over arbitrary fields whose elements are rational functions with no poles at infinity and prescribed finite poles. Complete systems of invariants are provided for each one of these equivalence relations and the relationship between both systems of invariants is clarified. This result can be seen as an extension of the classical theorem on pole assignment by Rosenbrock.

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3-Algebras in String Theory

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1. Introduction

In this chapter, we review 3-algebras that appear as fundamental properties of string theory. 3-algebra is a generalization of Lie algebra; it is defined by a tri-linear bracket instead of by a bi-linear bracket, and satisfies fundamental identity, which is a generalization of Jacobi identity [1–3]. We consider 3-algebras equipped with invariant metrics in order to apply them to physics.

It has been expected that there exists M-theory, which unifies string theories. In M-theory, some structures of 3-algebras were found recently. First, it was found that by using $u(N) \oplus u(N)$ Hermitian 3-algebra, we can describe a low energy effective action of N coincident supermembranes [4–8], which are fundamental objects in M-theory.

With this as motivation, 3-algebras with invariant metrics were classified [9–22]. Lie 3-algebras are defined in real vector spaces and tri-linear brackets of them are totally anti-symmetric in all the three entries. Lie 3-algebras with invariant metrics are classified into $\mathcal{A}_4$ algebra, and Lorentzian Lie 3-algebras, which have metrics with indefinite signatures. On the other hand, Hermitian 3-algebras are defined in Hermitian vector spaces and their tri-linear brackets are complex linear and anti-symmetric in the first two entries, whereas complex anti-linear in the third entry. Hermitian 3-algebras with invariant metrics are classified into $u(N) \oplus u(M)$ and $sp(2N) \oplus u(1)$ Hermitian 3-algebras.

Moreover, recent studies have indicated that there also exist structures of 3-algebras in the Green-Schwartz supermembrane action, which defines full perturbative dynamics of a supermembrane. It had not been clear whether the total supermembrane action including fermions has structures of 3-algebras, whereas the bosonic part of the action can be described by using a tri-linear bracket, called Nambu bracket [23, 24], which is a generalization of Poisson bracket. If we fix to a light-cone gauge, the total action can be described by using Poisson bracket, that is, only structures of Lie algebra are left in this gauge [25]. However, it was shown under an approximation that the total action can be described by Nambu bracket if we fix to a semi-light-cone gauge [26]. In this gauge, the eleven dimensional space-time of M-theory is manifest in the supermembrane action, whereas only ten dimensional part is manifest in the light-cone gauge.

The BFSS matrix theory is conjectured to describe an infinite momentum frame (IMF) limit of M-theory [27] and many evidences were found. The action of the BFSS matrix theory can be obtained by replacing Poisson bracket with a finite dimensional Lie algebra's bracket in the supermembrane action in the light-cone gauge. Because of this structure, only variables that represent the ten dimensional part of the eleven-dimensional space-time are manifest in the BFSS matrix theory. Recently, 3-algebra models of M-theory were proposed [26, 28, 29], by replacing Nambu bracket with finite dimensional 3-algebras' brackets in an action that is shown, by using an approximation, to be equivalent to the semi-light-cone supermembrane action. All the variables that represent the eleven dimensional space-time are manifest in these models. It was shown that if the DLCQ limit of the 3-algebra models of M-theory is taken, they reduce to the BFSS matrix theory [26, 28], as they should [30–35].

2. Definition and classification of metric Hermitian 3-algebra

In this section, we will define and classify the Hermitian 3-algebras equipped with invariant metrics.

2.1. General structure of metric Hermitian 3-algebra

The metric Hermitian 3-algebra is a map $V \times V \times V \to V$ defined by $(x, y, z) \mapsto [x, y; z]$, where the 3-bracket is complex linear in the first two entries, whereas complex anti-linear in the last entry, equipped with a metric $< x, y >$, satisfying the following properties:
the fundamental identity

$$[[x, y; z], v; w] = [[x, v; w], y; z] + [x, [y, v; w]; z] - [x, y; [z, w; v]] \tag{1}$$

the metric invariance

$$< [x, v; w], y > - < x, [y, w; v] >= 0 \tag{2}$$

and the anti-symmetry

$$[x, y; z] = -[y, x; z] \tag{3}$$

for

$$x, y, z, v, w \in V \tag{4}$$

The Hermitian 3-algebra generates a symmetry, whose generators $D(x, y)$ are defined by

$$D(x, y)z := [z, x; y] \tag{5}$$

From (1), one can show that $D(x, y)$ form a Lie algebra,

$$[D(x, y), D(v, w)] = D(D(x, y)v, w) - D(v, D(y, x)w) \tag{6}$$

There is an one-to-one correspondence between the metric Hermitian 3-algebra and a class of metric complex super Lie algebras [19]. Such a class satisfies the following conditions among complex super Lie algebras $S = S_0 \oplus S_1$, where S_0 and S_1 are even and odd parts, respectively. S_1 is decomposed as $S_1 = V \oplus \bar{V}$, where V is an unitary representation of S_0: for $a \in S_0$, $u, v \in V$,

$$[a, u] \in V \tag{7}$$

and

$$< [a,u],v > + < u,[a^*,v] >= 0 \tag{8}$$

$\bar{v} \in \bar{V}$ is defined by

$$\bar{v} =< \ ,v > \tag{9}$$

The super Lie bracket satisfies

$$[V,V] = 0, \quad [\bar{V},\bar{V}] = 0 \tag{10}$$

From the metric Hermitian 3-algebra, we obtain the class of the metric complex super Lie algebra in the following way. The elements in S_0, V, and $\bar{V}$ are defined by (5), (4), and (9), respectively. The algebra is defined by (6) and

$$\begin{aligned}
&[D(x,y),z] := D(x,y)z = [z,x;y] \\
&[D(x,y),\bar{z}] := -D(\bar{y},x)z = -[z,\bar{y};x] \\
&[x,\bar{y}] := D(x,y) \\
&[x,y] := 0 \\
&[\bar{x},\bar{y}] := 0
\end{aligned} \tag{11}$$

One can show that this algebra satisfies the super Jacobi identity and (7)-(10) as in [19].

Inversely, from the class of the metric complex super Lie algebra, we obtain the metric Hermitian 3-algebra by

$$[x,y;z] := \alpha[[y,\bar{z}],x] \tag{12}$$

where α is an arbitrary constant. One can also show that this algebra satisfies (1)-(3) for (4) as in [19].

2.2. Classification of metric Hermitian 3-algebra

The classical Lie super algebras satisfying (7)-(10) are $A(m-1,n-1)$ and $C(n+1)$. The even parts of $A(m-1,n-1)$ and $C(n+1)$ are $u(m) \oplus u(n)$ and $sp(2n) \oplus u(1)$, respectively. Because the metric Hermitian 3-algebra one-to-one corresponds to this class of the super Lie algebra, the metric Hermitian 3-algebras are classified into $u(m) \oplus u(n)$ and $sp(2n) \oplus u(1)$ Hermitian 3-algebras.

First, we will construct the $u(m) \oplus u(n)$ Hermitian 3-algebra from $A(m-1,n-1)$, according to the relation in the previous subsection. $A(m-1,n-1)$ is simple and is obtained by dividing $sl(m,n)$ by its ideal. That is, $A(m-1,n-1) = sl(m,n)$ when $m \neq n$ and $A(n-1,n-1) = sl(n,n)/\lambda 1_{2n}$.

Real $sl(m,n)$ is defined by

$$\begin{pmatrix} h_1 & c \\ ic^\dagger & h_2 \end{pmatrix} \tag{13}$$

where h_1 and h_2 are $m \times m$ and $n \times n$ anti-Hermite matrices and c is an $n \times m$ arbitrary complex matrix. Complex $sl(m,n)$ is a complexification of real $sl(m,n)$, given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{14}$$

where α, β, γ, and δ are $m \times m$, $n \times m$, $m \times n$, and $n \times n$ complex matrices that satisfy

$$\mathrm{tr}\alpha = \mathrm{tr}\delta \tag{15}$$

Complex $A(m-1, n-1)$ is decomposed as $A(m-1, n-1) = S_0 \oplus V \oplus \bar{V}$, where

$$\begin{aligned}\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} &\in S_0 \\ \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} &\in V \\ \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} &\in \bar{V}\end{aligned} \tag{16}$$

(9) is rewritten as $V \to \bar{V}$ defined by

$$B = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \mapsto B^{\dagger} = \begin{pmatrix} 0 & 0 \\ \beta^{\dagger} & 0 \end{pmatrix} \tag{17}$$

where $B \in V$ and $B^{\dagger} \in \bar{V}$. (12) is rewritten as

$$[X, Y; Z] = \alpha[[Y, Z^{\dagger}], X] = \alpha \begin{pmatrix} 0 & yz^{\dagger}x - xz^{\dagger}y \\ 0 & 0 \end{pmatrix} \tag{18}$$

for

$$\begin{aligned} X &= \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in V \\ Y &= \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in V \\ Z &= \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \in V \end{aligned} \tag{19}$$

As a result, we obtain the $u(m) \oplus u(n)$ Hermitian 3-algebra,

$$[x, y; z] = \alpha(yz^{\dagger}x - xz^{\dagger}y) \tag{20}$$

where x, y, and z are arbitrary $n \times m$ complex matrices. This algebra was originally constructed in [8].

Inversely, from (20), we can construct complex $A(m-1, n-1)$. (5) is rewritten as

$$D(x, y) = (xy^{\dagger}, y^{\dagger}x) \in S_0 \tag{21}$$

(6) and (11) are rewritten as

$$\begin{aligned} &[(xy^{\dagger}, y^{\dagger}x), (x'y'^{\dagger}, y'^{\dagger}x')] = ([xy^{\dagger}, x'y'^{\dagger}], [y^{\dagger}x, y'^{\dagger}x']) \\ &[(xy^{\dagger}, y^{\dagger}x), z] = xy^{\dagger}z - zy^{\dagger}x \\ &[(xy^{\dagger}, y^{\dagger}x), w^{\dagger}] = y^{\dagger}xw^{\dagger} - w^{\dagger}xy^{\dagger} \\ &[x, y^{\dagger}] = (xy^{\dagger}, y^{\dagger}x) \\ &[x, y] = 0 \\ &[x^{\dagger}, y^{\dagger}] = 0 \end{aligned} \tag{22}$$

This algebra is summarized as

$$\left[\begin{pmatrix} xy^\dagger & z \\ w^\dagger & y^\dagger x \end{pmatrix}, \begin{pmatrix} x'y'^\dagger & z' \\ w'^\dagger & y'^\dagger x' \end{pmatrix}\right] \tag{23}$$

which forms complex $A(m-1, n-1)$.

Next, we will construct the $sp(2n) \oplus u(1)$ Hermitian 3-algebra from $C(n+1)$. Complex $C(n+1)$ is decomposed as $C(n+1) = S_0 \oplus V \oplus \bar{V}$. The elements are given by

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & -a^T \end{pmatrix} \in S_0$$
$$\begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix} \in V$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y_1 & y_2 \\ y_2^T & 0 & 0 & 0 \\ -y_1^T & 0 & 0 & 0 \end{pmatrix} \in \bar{V} \tag{24}$$

where α is a complex number, a is an arbitrary $n \times n$ complex matrix, b and c are $n \times n$ complex symmetric matrices, and x_1, x_2, y_1 and y_2 are $n \times 1$ complex matrices. (9) is rewritten as $V \to \bar{V}$ defined by $B \mapsto \bar{B} = UB^*U^{-1}$, where $B \in V$, $\bar{B} \in \bar{V}$ and

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & -\mathbf{1} & 0 \end{pmatrix} \tag{25}$$

Explicitly,

$$B = \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix} \mapsto \bar{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_2^* & -x_1^* \\ -x_1^\dagger & 0 & 0 & 0 \\ -x_2^\dagger & 0 & 0 & 0 \end{pmatrix} \tag{26}$$

(12) is rewritten as

$$\begin{aligned} [X, Y; Z] &:= \alpha[[Y, \bar{Z}], X] \\ &= \alpha \left[\left[\begin{pmatrix} 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & 0 \\ 0 & y_2^T & 0 & 0 \\ 0 & -y_1^T & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & z_2^* & -z_1^* \\ -z_1^\dagger & 0 & 0 & 0 \\ -z_2^\dagger & 0 & 0 & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix}\right] \\ &= \alpha \begin{pmatrix} 0 & 0 & w_1 & w_2 \\ 0 & 0 & 0 & 0 \\ 0 & w_2^T & 0 & 0 \\ 0 & -w_1^T & 0 & 0 \end{pmatrix} \end{aligned} \tag{27}$$

for

$$X = \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix} \in V$$

$$Y = \begin{pmatrix} 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & 0 \\ 0 & y_2^T & 0 & 0 \\ 0 & -y_1^T & 0 & 0 \end{pmatrix} \in V$$

$$Z = \begin{pmatrix} 0 & 0 & z_1 & z_2 \\ 0 & 0 & 0 & 0 \\ 0 & z_2^T & 0 & 0 \\ 0 & -z_1^T & 0 & 0 \end{pmatrix} \in V \tag{28}$$

where w_1 and w_2 are given by

$$(w_1, w_2) = -(y_1 z_1^\dagger + y_2 z_2^\dagger)(x_1, x_2) + (x_1 z_1^\dagger + x_2 z_2^\dagger)(y_1, y_2) + (x_2 y_1^T - x_1 y_2^T)(z_2^*, -z_1^*) \tag{29}$$

As a result, we obtain the $sp(2n) \oplus u(1)$ Hermitian 3-algebra,

$$[x, y; z] = \alpha((y \odot \tilde{z})x + (\tilde{z} \odot x)y - (x \odot y)\tilde{z}) \tag{30}$$

for $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$, where x_1, x_2, y_1, y_2, z_1, and z_2 are n-vectors and

$$\tilde{z} = (z_2^*, -z_1^*)$$
$$a \odot b = a_1 \cdot b_2 - a_2 \cdot b_1 \tag{31}$$

3. 3-algebra model of M-theory

In this section, we review the fact that the supermembrane action in a semi-light-cone gauge can be described by Nambu bracket, where structures of 3-algebra are manifest. The 3-algebra Models of M-theory are defined based on the semi-light-cone supermembrane action. We also review that the models reduce to the BFSS matrix theory in the DLCQ limit.

3.1. Supermembrane and 3-algebra model of M-theory

The fundamental degrees of freedom in M-theory are supermembranes. The action of the covariant supermembrane action in M-theory [36] is given by

$$S_{M2} = \int d^3\sigma \Big(\sqrt{-G} + \frac{i}{4}\epsilon^{\alpha\beta\gamma}\bar{\Psi}\Gamma_{MN}\partial_\alpha\Psi(\Pi_\beta{}^M\Pi_\gamma{}^N + \frac{i}{2}\Pi_\beta{}^M\bar{\Psi}\Gamma^N\partial_\gamma\Psi$$
$$-\frac{1}{12}\bar{\Psi}\Gamma^M\partial_\beta\Psi\bar{\Psi}\Gamma^N\partial_\gamma\Psi)\Big) \tag{32}$$

where $M, N = 0, \cdots, 10$, $\alpha, \beta, \gamma = 0, 1, 2$, $G_{\alpha\beta} = \Pi_\alpha{}^M\Pi_{\beta M}$ and $\Pi_\alpha{}^M = \partial_\alpha X^M - \frac{i}{2}\bar{\Psi}\Gamma^M\partial_\alpha\Psi$. Ψ is a $SO(1,10)$ Majorana fermion.

This action is invariant under dynamical supertransformations,

$$\begin{aligned}\delta\Psi &= \epsilon \\ \delta X^M &= -i\bar{\Psi}\Gamma^M\epsilon\end{aligned} \tag{33}$$

These transformations form the $\mathcal{N}=1$ supersymmetry algebra in eleven dimensions,

$$[\delta_1, \delta_2]X^M = -2i\epsilon_1\Gamma^M\epsilon_2$$

$$[\delta_1, \delta_2]\Psi = 0 \tag{34}$$

The action is also invariant under the κ-symmetry transformations,

$$\begin{aligned}\delta\Psi &= (1+\Gamma)\kappa(\sigma) \\ \delta X^M &= i\bar{\Psi}\Gamma^M(1+\Gamma)\kappa(\sigma)\end{aligned} \tag{35}$$

where

$$\Gamma = \frac{1}{3!\sqrt{-G}}\epsilon^{\alpha\beta\gamma}\Pi^L_\alpha\Pi^M_\beta\Pi^N_\gamma\Gamma_{LMN} \tag{36}$$

If we fix the κ-symmetry (35) of the action by taking a semi-light-cone gauge [26][1]

$$\Gamma^{012}\Psi = -\Psi \tag{37}$$

we obtain a semi-light-cone supermembrane action,

$$\begin{aligned}S_{M2} = \int d^3\sigma\Big(\sqrt{-G} + \frac{i}{4}\epsilon^{\alpha\beta\gamma}\big(\bar{\Psi}\Gamma_{\mu\nu}\partial_\alpha\Psi(\Pi_\beta{}^\mu\Pi_\gamma{}^\nu + \frac{i}{2}\Pi_\beta{}^\mu\bar{\Psi}\Gamma^\nu\partial_\gamma\Psi - \frac{1}{12}\bar{\Psi}\Gamma^\mu\partial_\beta\Psi\bar{\Psi}\Gamma^\nu\partial_\gamma\Psi) \\ +\bar{\Psi}\Gamma_{IJ}\partial_\alpha\Psi\partial_\beta X^I\partial_\gamma X^J\big)\Big)\end{aligned} \tag{38}$$

where $G_{\alpha\beta} = h_{\alpha\beta} + \Pi_\alpha{}^\mu\Pi_{\beta\mu}$, $\Pi_\alpha{}^\mu = \partial_\alpha X^\mu - \frac{i}{2}\bar{\Psi}\Gamma^\mu\partial_\alpha\Psi$, and $h_{\alpha\beta} = \partial_\alpha X^I\partial_\beta X_I$.

In [26], it is shown under an approximation up to the quadratic order in $\partial_\alpha X^\mu$ and $\partial_\alpha\Psi$ but exactly in X^I, that this action is equivalent to the continuum action of the 3-algebra model of M-theory,

$$\begin{aligned}S_{cl} = \int d^3\sigma\sqrt{-g}\Big(&-\frac{1}{12}\{X^I, X^J, X^K\}^2 - \frac{1}{2}(A_{\mu ab}\{\varphi^a, \varphi^b, X^I\})^2 \\ &-\frac{1}{3}E^{\mu\nu\lambda}A_{\mu ab}A_{\nu cd}A_{\lambda ef}\{\varphi^a, \varphi^c, \varphi^d\}\{\varphi^b, \varphi^e, \varphi^f\} + \frac{1}{2}\Lambda \\ &-\frac{i}{2}\bar{\Psi}\Gamma^\mu A_{\mu ab}\{\varphi^a, \varphi^b, \Psi\} + \frac{i}{4}\bar{\Psi}\Gamma_{IJ}\{X^I, X^J, \Psi\}\Big)\end{aligned} \tag{39}$$

where $I, J, K = 3, \cdots, 10$ and $\{\varphi^a, \varphi^b, \varphi^c\} = \epsilon^{\alpha\beta\gamma}\partial_\alpha\varphi^a\partial_\beta\varphi^b\partial_\gamma\varphi^c$ is the Nambu-Poisson bracket. An invariant symmetric bilinear form is defined by $\int d^3\sigma\sqrt{-g}\varphi^a\varphi^b$ for complete basis φ^a in three dimensions. Thus, this action is manifestly VPD covariant even when the world-volume metric is flat. X^I is a scalar and Ψ is a $SO(1,2)\times SO(8)$ Majorana-Weyl fermion

[1] Advantages of a semi-light-cone gauges against a light-cone gauge are shown in [37–39]

satisfying (37). $E^{\mu\nu\lambda}$ is a Levi-Civita symbol in three dimensions and Λ is a cosmological constant.

The continuum action of 3-algebra model of M-theory (39) is invariant under 16 dynamical supersymmetry transformations,

$$\begin{aligned}
&\delta X^I = i\bar{\epsilon}\Gamma^I\Psi \\
&\delta A_\mu(\sigma,\sigma') = \frac{i}{2}\bar{\epsilon}\Gamma_\mu\Gamma_I(X^I(\sigma)\Psi(\sigma') - X^I(\sigma')\Psi(\sigma)), \\
&\delta\Psi = -A_{\mu ab}\{\varphi^a,\varphi^b,X^I\}\Gamma^\mu\Gamma_I\epsilon - \frac{1}{6}\{X^I,X^J,X^K\}\Gamma_{IJK}\epsilon
\end{aligned} \tag{40}$$

where $\Gamma_{012}\epsilon = -\epsilon$. These supersymmetries close into gauge transformations on-shell,

$$\begin{aligned}
&[\delta_1,\delta_2]X^I = \Lambda_{cd}\{\varphi^c,\varphi^d,X^I\} \\
&[\delta_1,\delta_2]A_{\mu ab}\{\varphi^a,\varphi^b,\quad\} = \Lambda_{ab}\{\varphi^a,\varphi^b,A_{\mu cd}\{\varphi^c,\varphi^d,\quad\}\} \\
&\qquad -A_{\mu ab}\{\varphi^a,\varphi^b,\Lambda_{cd}\{\varphi^c,\varphi^d,\quad\}\} + 2i\bar{\epsilon}_2\Gamma^\nu\epsilon_1 O^A_{\mu\nu} \\
&[\delta_1,\delta_2]\Psi = \Lambda_{cd}\{\varphi^c,\varphi^d,\Psi\} + (i\bar{\epsilon}_2\Gamma^\mu\epsilon_1\Gamma_\mu - \frac{i}{4}\bar{\epsilon}_2\Gamma^{KL}\epsilon_1\Gamma_{KL})O^\Psi
\end{aligned} \tag{41}$$

where gauge parameters are given by $\Lambda_{ab} = 2i\bar{\epsilon}_2\Gamma^\mu\epsilon_1 A_{\mu ab} - i\bar{\epsilon}_2\Gamma_{JK}\epsilon_1 X^J_a X^K_b$. $O^A_{\mu\nu} = 0$ and $O^\Psi = 0$ are equations of motions of $A_{\mu\nu}$ and Ψ, respectively, where

$$\begin{aligned}
O^A_{\mu\nu} &= A_{\mu ab}\{\varphi^a,\varphi^b,A_{\nu cd}\{\varphi^c,\varphi^d,\quad\}\} - A_{\nu ab}\{\varphi^a,\varphi^b,A_{\mu cd}\{\varphi^c,\varphi^d,\quad\}\} \\
&\quad +E_{\mu\nu\lambda}(-\{X^I,A^\lambda_{ab}\{\varphi^a,\varphi^b,X_I\},\quad\} + \frac{i}{2}\{\bar{\Psi},\Gamma^\lambda\Psi,\quad\}) \\
O^\Psi &= -\Gamma^\mu A_{\mu ab}\{\varphi^a,\varphi^b,\Psi\} + \frac{1}{2}\Gamma_{IJ}\{X^I,X^J,\Psi\}
\end{aligned} \tag{42}$$

(41) implies that a commutation relation between the dynamical supersymmetry transformations is

$$\delta_2\delta_1 - \delta_1\delta_2 = 0 \tag{43}$$

up to the equations of motions and the gauge transformations.

This action is invariant under a translation,

$$\delta X^I(\sigma) = \eta^I, \qquad \delta A^\mu(\sigma,\sigma') = \eta^\mu(\sigma) - \eta^\mu(\sigma') \tag{44}$$

where η^I are constants.

The action is also invariant under 16 kinematical supersymmetry transformations

$$\tilde{\delta}\Psi = \tilde{\epsilon} \tag{45}$$

and the other fields are not transformed. $\tilde{\epsilon}$ is a constant and satisfy $\Gamma_{012}\tilde{\epsilon} = \tilde{\epsilon}$. $\tilde{\epsilon}$ and ϵ should come from sixteen components of thirty-two $\mathcal{N} = 1$ supersymmetry parameters in eleven dimensions, corresponding to eigen values ± 1 of Γ_{012}, respectively. This $\mathcal{N} = 1$ supersymmetry consists of remaining 16 target-space supersymmetries and transmuted 16 κ-symmetries in the semi-light-cone gauge [25, 26, 40].

A commutation relation between the kinematical supersymmetry transformations is given by

$$\tilde{\delta}_2\tilde{\delta}_1 - \tilde{\delta}_1\tilde{\delta}_2 = 0 \tag{46}$$

A commutator of dynamical supersymmetry transformations and kinematical ones acts as

$$\begin{aligned}(\tilde{\delta}_2\delta_1 - \delta_1\tilde{\delta}_2)X^I(\sigma) &= i\bar{\epsilon}_1\Gamma^I\tilde{\epsilon}_2 \equiv \eta_0^I \\ (\tilde{\delta}_2\delta_1 - \delta_1\tilde{\delta}_2)A^\mu(\sigma,\sigma') &= \frac{i}{2}\bar{\epsilon}_1\Gamma^\mu\Gamma_I(X^I(\sigma) - X^I(\sigma'))\tilde{\epsilon}_2 \equiv \eta_0^\mu(\sigma) - \eta_0^\mu(\sigma')\end{aligned} \tag{47}$$

where the commutator that acts on the other fields vanishes. Thus, the commutation relation is given by

$$\tilde{\delta}_2\delta_1 - \delta_1\tilde{\delta}_2 = \delta_\eta \tag{48}$$

where δ_η is a translation.

If we change a basis of the supersymmetry transformations as

$$\begin{aligned}\delta' &= \delta + \tilde{\delta} \\ \tilde{\delta}' &= i(\delta - \tilde{\delta})\end{aligned} \tag{49}$$

we obtain

$$\begin{aligned}\delta_2'\delta_1' - \delta_1'\delta_2' &= \delta_\eta \\ \tilde{\delta}_2'\tilde{\delta}_1' - \tilde{\delta}_1'\tilde{\delta}_2' &= \delta_\eta \\ \tilde{\delta}_2'\delta_1' - \delta_1'\tilde{\delta}_2' &= 0\end{aligned} \tag{50}$$

These thirty-two supersymmetry transformations are summarised as $\Delta = (\delta', \tilde{\delta}')$ and (50) implies the $\mathcal{N} = 1$ supersymmetry algebra in eleven dimensions,

$$\Delta_2\Delta_1 - \Delta_1\Delta_2 = \delta_\eta \tag{51}$$

3.2. Lie 3-algebra models of M-theory

In this and next subsection, we perform the second quantization on the continuum action of the 3-algebra model of M-theory: By replacing the Nambu-Poisson bracket in the action (39) with brackets of finite-dimensional 3-algebras, Lie and Hermitian 3-algebras, we obtain the Lie and Hermitian 3-algebra models of M-theory [26, 28], respectively. In this section, we review the Lie 3-algebra model.

If we replace the Nambu-Poisson bracket in the action (39) with a completely antisymmetric real 3-algebra's bracket [21, 22],

$$\begin{aligned}\int d^3\sigma\sqrt{-g} &\to \langle\quad\rangle \\ \{\varphi^a, \varphi^b, \varphi^c\} &\to [T^a, T^b, T^c]\end{aligned} \tag{52}$$

we obtain the Lie 3-algebra model of M-theory [26, 28],

$$\begin{aligned}S_0 = \Big\langle &-\frac{1}{12}[X^I, X^J, X^K]^2 - \frac{1}{2}(A_{\mu ab}[T^a, T^b, X^I])^2 \\ &-\frac{1}{3}E^{\mu\nu\lambda}A_{\mu ab}A_{\nu cd}A_{\lambda ef}[T^a, T^c, T^d][T^b, T^e, T^f] \\ &-\frac{i}{2}\bar{\Psi}\Gamma^\mu A_{\mu ab}[T^a, T^b, \Psi] + \frac{i}{4}\bar{\Psi}\Gamma_{IJ}[X^I, X^J, \Psi]\Big\rangle\end{aligned} \tag{53}$$