A Radical Approach to Lebesgue's Theory of Integration

David M. Bressoud

A RADICAL APPROACH TO LEBESGUE'S THEORY OF INTEGRATION

DAVID M. BRESSOUD
Macalester College

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press
32 Avenue of the Americas, New York, NY 10013-2473, USA

www.cambridge.org
Information on this title: www.cambridge.org/9780521884747

The Mathematical Association of America
1529 Eighteenth Street, NW, Washington, DC 20036

First published 2008

Printed in the United States of America

A catalog record for this publication is available from the British Library.

Library of Congress Cataloging in Publication Data

Bressoud, David M., 1950–
A Radical approach to Lebesgue's theory of integration / David M. Bressoud.
p. cm. – (Classroom resource materials)
Includes bibliographical references and index.
ISBN-13: 978-0-521-88474-7 (hardback)
ISBN-10: 0-521-88474-8 (hardback)
ISBN-13: 978-0-521-71183-8 (pbk.)
ISBN-10: 0-521-71183-5 (pbk.)
1. Integrals, Generalized. I. Title. II. Series.
QA312.B67 2008
515′.42–dc22 2007035326

ISBN 978-0-521-88474-7 hardback
ISBN 978-0-521-71183-8 paperback

A RADICAL APPROACH TO LEBESGUE'S THEORY OF INTEGRATION

Meant for advanced undergraduate and graduate students in mathematics, this lively introduction to measure theory and Lebesgue integration is rooted in and motivated by the historical questions that led to its development. The author stresses the original purpose of the definitions and theorems and highlights some of the difficulties that were encountered as these ideas were refined.

The story begins with Riemann's definition of the integral, a definition created so that he could understand how broadly one could define a function and yet have it be integrable. The reader then follows the efforts of many mathematicians who wrestled with the difficulties inherent in the Riemann integral, leading to the work in the late nineteenth and early twentieth centuries of Jordan, Borel, and Lebesgue, who finally broke with Riemann's definition. Ushering in a new way of understanding integration, they opened the door to fresh and productive approaches to many of the previously intractable problems of analysis.

David M. Bressoud is the DeWitt Wallace Professor of Mathematics at Macalester College. He was a Peace Corps Volunteer in Antigua, West Indies, received his PhD from Temple University, and taught at The Pennsylvania State University before moving to Macalester. He has held visiting positions at the Institute for Advanced Study, the University of Wisconsin, the University of Minnesota, and the University of Strasbourg. He has received a Sloan Fellowship, a Fulbright Fellowship, and the MAA Distinguished Teaching Award. He has published more than 50 research articles in number theory, partition theory, combinatorics, and the theory of special functions. His other books include *Factorization and Primality Testing*, *Second Year Calculus from Celestial Mechanics to Special Relativity*, *A Radical Approach to Real Analysis*, and *Proofs and Confirmations*, for which he won the MAA Beckenbach Book Prize.

MAA TEXTBOOKS

Combinatorics: A Problem Oriented Approach, Daniel A. Marcus
Complex Numbers and Geometry, Liang-shin Hahn
A Course in Mathematical Modeling, Douglas Mooney and Randall Swift
Creative Mathematics, H. S. Wall
Cryptological Mathematics, Robert Edward Lewand
Differential Geometry and Its Applications, John Oprea
Elementary Cryptanalysis, Abraham Sinkov
Elementary Mathematical Models, Dan Kalman
Essentials of Mathematics, Margie Hale
Field Theory and Its Classical Problems, Charles Hadlock
Fourier Series, Rajendra Bhatia
Game Theory and Strategy, Philip D. Straffin
Geometry Revisited, H. S. M. Coxeter and S. L. Greitzer
Knot Theory, Charles Livingston
Mathematical Connections: A Companion for Teachers and Others, Al Cuoco
Mathematical Modeling in the Environment, Charles Hadlock
Mathematics for Business Decisions Part 1: Probability and Simulation (electronic textbook), Richard B. Thompson and Christopher G. Lamoureux
Mathematics for Business Decisions Part 2: Calculus and Optimization (electronic textbook), Richard B. Thompson and Christopher G. Lamoureux
The Mathematics of Games and Gambling, Edward Packel
Math Through the Ages, William Berlinghoff and Fernando Gouvea
Noncommutative Rings, I. N. Herstein
Non-Euclidean Geometry, H. S. M. Coxeter
Number Theory Through Inquiry, David C. Marshall, Edward Odell, and Michael Starbird
A Primer of Real Functions, Ralph P. Boas

A Radical Approach to Real Analysis, 2nd edition, David M. Bressoud
Real Infinite Series, Daniel D. Bonar and Michael Khoury, Jr.
Topology Now!, Robert Messer and Philip Straffin
Understanding Our Quantitative World, Janet Andersen and Todd Swanson

MAA Service Center
P.O. Box 91112
Washington, DC 20090-1112
1-800-331-1MAA FAX: 1-301-206-9789

Preface

I look at the burning question of the foundations of infinitesimal analysis without sorrow, anger, or irritation. What Weierstrass – Cantor – did was *very good. That's the way it* had *to be done. But whether this corresponds to what is in the depths of our consciousness is a very different question. I cannot but see a stark contradiction between the intuitively clear fundamental formulas of the integral calculus and the incomparably artificial and complex work of the "justification" and their "proofs." One must be quite stupid not to see this at once, and quite careless if, after having seen this, one can get used to this artificial, logical atmosphere, and can later on forget this stark contradiction.*

– Nikolaĭ Nikolaevich Luzin

Nikolaĭ Luzin reminds us of a truth too often forgotten in the teaching of analysis; the ideas, methods, definitions, and theorems of this study are neither natural nor intuitive. It is all too common for students to emerge from this study with little sense of how the concepts and results that constitute modern analysis hang together. Here more than anywhere else in the advanced undergraduate/beginning graduate curriculum, the historical context is critical to developing an understanding of the mathematics.

This historical context is both interesting and pedagogically informative. From transfinite numbers to the Heine–Borel theorem to Lebesgue measure, these ideas arose from practical problems but were greeted with a skepticism that betrayed confusion. Understanding what they mean and how they can be used was an uncertain process. We should expect our students to encounter difficulties at precisely those points at which the contemporaries of Weierstrass, Cantor, and Lebesgue had balked.

Throughout this text I have tried to emphasize that no one set out to invent measure theory or functional analysis. I find it both surprising and immensely satisfying that the search for understanding of Fourier series continued to be one of the principal driving forces behind the development of analysis well into the twentieth century. The tools that these mathematicians had at hand were not adequate to the task. In particular, the Riemann integral was poorly adapted to their needs.

It took several decades of wrestling with frustrating difficulties before mathematicians were willing to abandon the Riemann integral. The route to its eventual replacement, the Lebesgue integral, led through a sequence of remarkable insights into the complexities of the real number line. By the end of 1890s, it was recognized that analysis and the study of sets were inextricably linked. From this rich interplay, measure theory would emerge. With it came what today we call Lebesgue's dominated convergence theorem, the holy grail of nineteenth-century analysis. What so many had struggled so hard to discover now appeared as a gift that was almost free.

This text is an introduction to measure theory and Lebesgue integration, though anyone using it to support such a course must be forewarned that I have intentionally avoided stating results in their greatest possible generality. Almost all results are given only for the real number line. Theorems that are true over any compact set are often stated only for closed, bounded intervals. I want students to get a feel for these results, what they say, and why they are important. Close examination of the most general conditions under which conclusions will hold is something that can come later, if and when it is needed.

The title of this book was chosen to communicate two important points. First, this is a sequel to *A Radical Approach to Real Analysis* (ARATRA). That book ended with Riemann's definition of the integral. That is where this text begins. All of the topics that one might expect to find in an undergraduate analysis book that were not in ARATRA are contained here, including the topology of the real number line, fundamentals of set theory, transfinite cardinals, the Bolzano–Weierstrass theorem, and the Heine–Borel theorem. I did not include them in the first volume because I felt I could not do them justice there and because, historically, they are quite sophisticated insights that did not arise until the second half of the nineteenth century.

Second, this book owes a tremendous debt to Thomas Hawkins' *Lebesgue's Theory of Integration: Its Origins and Development*. Like ARATRA, this book is not intended to be read as a history of the development of analysis. Rather, this is a textbook informed by history, attempting to communicate the motivations, uncertainties, and difficulties surrounding the key concepts. This task would have been far more difficult without Hawkins as a guide. Those who are intrigued by the historical details encountered in this book are encouraged to turn to Hawkins and other historians of this period for fuller explanation.

Even more than ARATRA, this is the story of many contributions by many members of a large community of mathematicians working on different pieces of the puzzle. I hope that I have succeeded in opening a small window into the workings of this community. One of the most intriguing of these mathematicians is Axel

Harnack, who keeps reappearing in our story because he kept making mistakes, but they were *good* mistakes. Harnack's errors condensed and made explicit many of the misconceptions of his time, and so helped others to find the correct path. For ARATRA, it was easy to select the four mathematicians who should grace the cover: Fourier, Cauchy, Abel, and Dirichlet stand out as those who shaped the origins of modern analysis. For this book, the choice is far less clear. Certainly I need to include Riemann and Lebesgue, for they initiate and bring to conclusion the principal elements of this story. Weierstrass? He trained and inspired the generation that would grapple with Riemann's work, but his contributions are less direct. Heine, du Bois-Reymond, Jordan, Hankel, Darboux, or Dini? They all made substantial progress toward the ultimate solution, but none of them stands out sufficiently. Cantor? Certainly yes. It was his recognition that set theory lies at the heart of analysis that would enable the progress of the next generation. Who should we select from that next generation: Peano, Volterra, Borel, Baire? Maybe Riesz or one of the others who built on Lebesgue's insights, bringing them to fruition? Now the choice is even less clear. I have settled on Borel for his impact as a young mathematician and to honor him as the true source of the Heine–Borel theorem, a result that I have been very tempted to refer to as he did: the first fundamental theorem of measure theory.

I have drawn freely on the scholarship of others. I must pay special tribute to Soo Bong Chae's *Lebesgue Integration*. When I first saw this book, my reaction was that I did not need to write my own on Lebesgue integration. Here was someone who had already put the subject into historical context, writing in an elegant yet accessible style. However, as I have used his book over the years, I have found that there is much that he leaves unsaid, and I disagree with his choice to use Riesz's approach to the Lebesgue integral, building it via an analysis of step functions. Riesz found an elegant route to Lebesgue integration, but in defining the integral first and using it to define Lebesgue measure, the motivation for developing these concepts is lost. Despite such fundamental divergences, the attentive reader will discover many close parallels between Chae's treatment and mine.

I am indebted to many people who read and commented on early drafts of this book. I especially thank Dave Renfro who gave generously of his time to correct many of my historical and mathematical errors. Steve Greenfield had the temerity to be the very first reader of my very first draft, and I appreciate his many helpful suggestions on the organization and presentation of this book. I also want to single out my students who, during the spring semester of 2007, struggled through a preliminary draft of this book and helped me in many ways to correct errors and improve the presentation of this material. They are Jacob Bond, Kyle Braam, Pawan Dhir, Elizabeth Gillaspy, Dan Gusset, Sam Handler, Kassa Haileyesus, Xi Luo, Jake Norton, Stella Stamenova, and Linh To.

I am also grateful to the mathematicians and historians of mathematics who suggested corrections and changes or helped me find information. These include Roger Cooke, Larry D'Antonio, Ivor Grattan-Guinness, Daesuk Han, Tom Hawkins, Mark Huibregtse, Nicholas Rose, Peter Ross, Jim Smoak, John Stillwell, and Sergio B. Volchan. I am also indebted to Don Albers of the MAA and Lauren Cowles of Cambridge who so enthusiastically embraced this project, and to the reviewers for both MAA and Cambridge whose names are unknown to me but who gave much good advice.

Corrections, commentary, and additional material for this book can be found at `www.macalester.edu/aratra`.

David M. Bressoud
bressoud@macalester.edu
June 19, 2007

Contents

1

Introduction

By 1850, most mathematicians thought they understood calculus. Real progress was being made in extending the tools of calculus to complex numbers and spaces of higher dimensions. Equipped with appropriate generalizations of Fourier series, solutions to partial differential equations were being found. Cauchy's insights had been assimilated, and the concepts that had been unclear during his pioneering work of the 1820s, concepts such as uniform convergence and uniform continuity, were coming to be understood. There was reason to feel confident.

One of the small, nagging problems that remained was the question of the convergence of the Fourier series expansion. When does it converge? When it does, can we be certain that it converges to the original function from which the Fourier coefficients were derived? In 1829, Peter Gustav Lejeune Dirichlet had proven that as long as a function is piecewise monotonic on a closed and bounded interval, the Fourier series converges to the original function. Dirichlet believed that functions did not have to be piecewise monotonic in order for the Fourier series to converge to the original function, but neither he nor anyone else had been able to weaken this assumption.

In the early 1850s, Bernard Riemann, a young protegé of Dirichlet and a student of Gauss, would make substantial progress in extending our understanding of trigonometric series. In so doing, the certainties of calculus would come into question. Over the next 60 years, five big questions would emerge and be answered. The answers would be totally unexpected. They would forever change the nature of analysis.

1. **When does a function have a Fourier series expansion that converges to that function?**
2. **What is integration?**
3. **What is the relationship between integration and differentiation?**

4. What is the relationship between continuity and differentiability?
5. When can an infinite series be integrated by integrating each term?

This book is devoted to explaining the answers to these five questions – answers that are very much intertwined. Before we tackle what happened after 1850, we need to understand what was known or believed in that year.

1.1 The Five Big Questions

Fourier Series

Fourier's method for expanding an arbitrary function F defined on $[-\pi, \pi]$ into a trigonometric series is to use integration to calculate coefficients:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos(kx)\, dx \quad (k \geq 0), \tag{1.1}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin(kx)\, dx \quad (k \geq 1). \tag{1.2}$$

The Fourier expansion is then given by

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(kx) + b_k \sin(kx)\right]. \tag{1.3}$$

The heuristic argument for the validity of this procedure is that if F really can be expanded in a series of the form given in Equation (1.3), then

$$\begin{aligned} &\int_{-\pi}^{\pi} F(x) \cos(nx)\, dx \\ &\quad = \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(kx) + b_k \sin(kx)\right] \right) \cos(nx)\, dx \\ &\quad = \int_{-\pi}^{\pi} \frac{a_0}{2} \cos(nx)\, dx + \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} a_k \cos(kx) \cos(nx)\, dx \\ &\qquad + \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} b_k \sin(kx) \cos(nx)\, dx. \end{aligned} \tag{1.4}$$

Since n and k are integers, all of the integrals are zero except for the one involving a_n. These integrals are easily evaluated:

$$\int_{-\pi}^{\pi} F(x) \cos(nx)\, dx = \pi a_n. \tag{1.5}$$

Similarly,

$$\int_{-\pi}^{\pi} F(x)\sin(nx)\,dx = \pi b_n. \tag{1.6}$$

This is a convincing heuristic, but it ignores the problem of interchanging integration and summation, and it sidesteps two crucial questions:

1. Are the integrals that produce the Fourier coefficients well-defined?
2. If these integrals can be evaluated, does the resulting Fourier series actually converge to the original function?

Not all functions are integrable. In the 1820s, Dirichlet proposed the following example.

Example 1.1. The **characteristic function of the rationals** is defined as

$$f(x) = \begin{cases} 1, & x \text{ is rational}, \\ 0, & x \text{ is not rational}. \end{cases}$$

This example demonstrates how very strange functions can be if we take seriously the definition of a function as a well-defined rule that assigns a value to each number in the domain. Dirichlet's example represents an important step in the evolution of the concept of function. To the early explorers of calculus, a function was an algebraic rule such as $\sin x$ or $x^2 - 3$, an expression that could be computed to whatever accuracy one might desire.

When Augustin-Louis Cauchy showed that any piecewise continuous function is integrable, he cemented the realization that functions could also be purely geometric, representable only as curves. Even in a situation in which a function has no explicit algebraic formulation, it is possible to make sense of its integral, provided the function is continuous.

Dirichlet stretched the concept of function to that of a rule that can be individually defined for each value of the domain. Once this conception of function is accepted, the gates are opened to very strange functions. At the very least, integrability can no longer be assumed.

The next problem is to show that our trigonometric series converges. In his 1829 paper, Dirichlet accomplished this, but he needed the hypothesis that the original function F is piecewise monotonic, that is the domain can be partitioned into a finite number of subintervals so that F is either monotonically increasing or monotonically decreasing on each subinterval.

The final question is whether the function to which it converges is the function F with which we started. Under the same assumptions, Dirichlet was able to show that this is the case, provided that at any points of discontinuity of F, the value taken by the function is the average of the limit from the left and the limit from the right.

Dirichlet's result implies that the functions one is likely to encounter in physical situations present no problems for conversion into Fourier series. Riemann recognized that it was important to be able to extend this technique to more complicated functions now arising in questions in number theory and geometry. The first step was to get a better handle on what we mean by integration.

Integration

It is ironic that integration took so long to get right because it is so much older than any other piece of calculus. Its roots lie in methods of calculating areas, volumes, and moments that were undertaken by such scientists as Archimedes (287–212 BC), Liu Hui (late third century AD), ibn al-Haytham (965–1039), and Johannes Kepler (1571–1630). The basic idea was always the same. To evaluate an area, one divided it into rectangles or triangles or other shapes of known area that together approximated the desired region. As more and smaller figures were used, the region would be matched more precisely. Some sort of limiting argument would then be invoked, some means of finding the actual area based on an analysis of the areas of the approximating regions.

Into the eighteenth century, integration was identified with the problem of "quadrature," literally the process of finding a square equal in area to a given area and thus, in practice, the problem of computing areas. In section 1 of Book I of his *Mathematical Principles of Natural Philosophy*, Newton explains how to calculate areas under curves. He gives a procedure that looks very much like the definition of the Riemann integral, and he justifies it by an argument that would be appropriate for any modern textbook.

Specifically, Newton begins by approximating the area under a decreasing curve by subdividing the domain into equal subintervals (see Figure 1.1). Above each subinterval, he constructs two rectangles: one whose height is the maximum value of the function on that interval (the circumscribed rectangle) and the other whose height is the minimum value of the function (the inscribed rectangle). The true area lies between the sum of the areas of the circumscribed rectangles and the sum of the areas of the inscribed rectangles.

The difference between these areas is the sum of the areas of the rectangles $aKbl$, $bLcm$, $cMdn$, $dDEo$. If we slide all of these rectangles to line up under $aKbl$, we see that the sum of their areas is just the change in height of the function multiplied by the length of any one subinterval. As we take narrower subintervals, the difference in the areas approaches zero. As Newton asserts: "The ultimate ratios which the inscribed figure, the circumscribed figure, and the curvilinear figure have

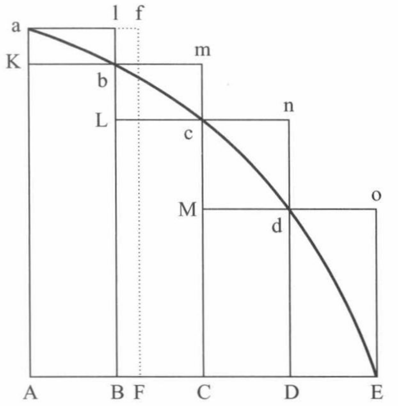

Figure 1.1. Newton's illustration from *Mathematical Principles of Natural Philosophy*. (Newton, 1999, p. 433)

to one another are ratios of equality," which is his way of saying that the ratio of any two of these areas approaches 1. Therefore, the areas are all approaching the same value as the length of the subinterval approaches 0.

In Lemma 3 of his book, Newton considers the case where the subintervals are not of equal length (using the dotted line fF in Figure 1.1 in place of lB). He observes that the sum of the differences of the areas is still less than the change in height multiplied by the length of the longest subinterval. We therefore get the same limit for the ratio so long as the length of the longest subinterval is approaching zero.

This method of finding areas is paradigmatic for an entire class of problems in which one is multiplying two quantities such as

- area = height × width,
- volume = cross-sectional area × width,
- moment = mass × distance,
- work = force × distance,
- distance = speed × time, or
- velocity = acceleration × time,

where the value of the first quantity can vary as the second quantity increases. For example, knowing that "distance = speed × time," we can find the distance traveled by a particle whose speed is a function of time, say $v(t) = 8t + 5, 0 \leq t \leq 4$. If we split the time into four intervals and use the velocity at the start of each interval, we get an approximation to the total distance:

$$\text{distance} \approx 5 \cdot 1 + 13 \cdot 1 + 21 \cdot 1 + 29 \cdot 1 = 68.$$