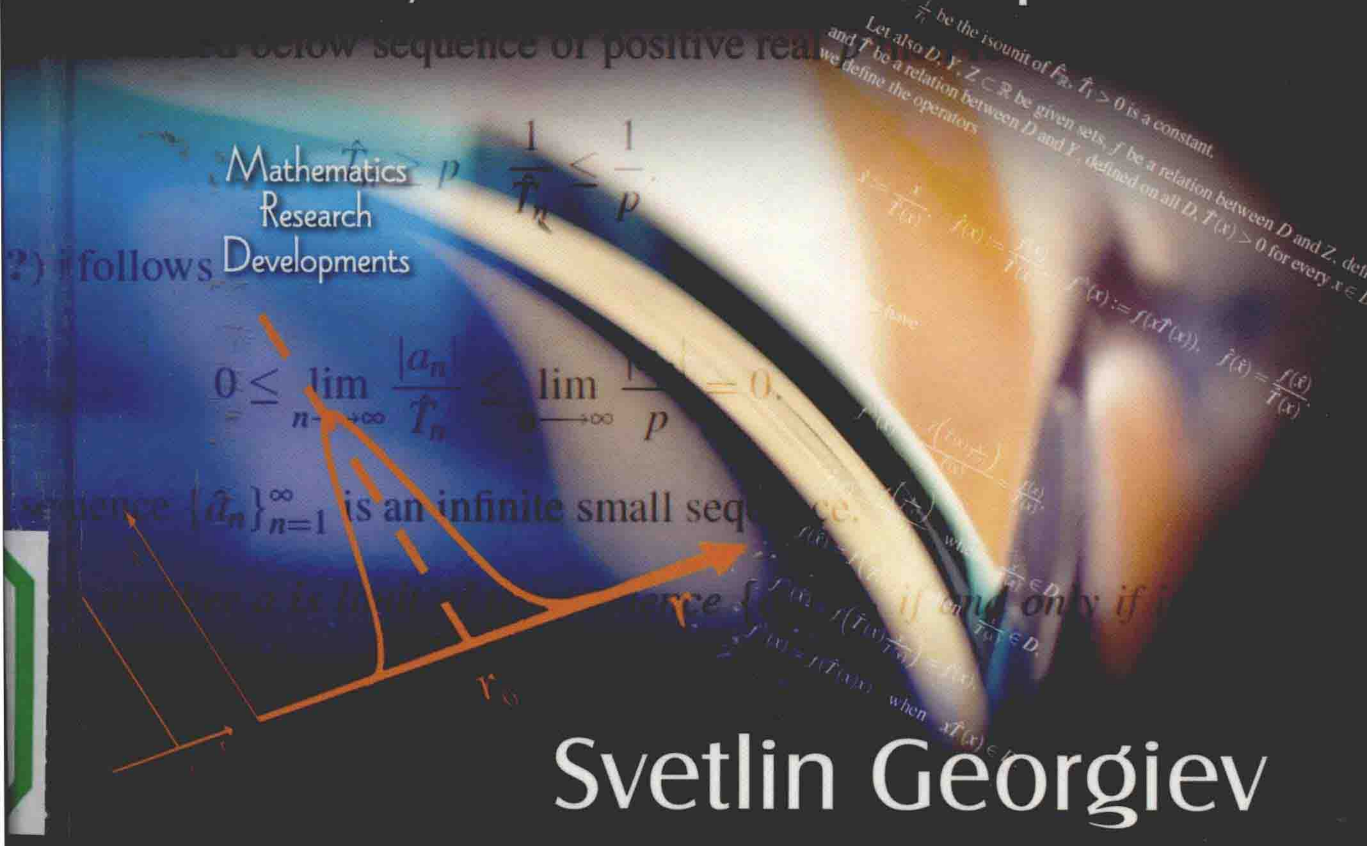
Volume 5

Foundations of Iso-Differential Calculus

Ordinary Iso-Differential Equations



Svetlin Georgiev

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MATHEMATICS RESEARCH DEVELOPMENTS

FOUNDATIONS OF ISO-DIFFERENTIAL CALCULUS

VOLUME 5
ISO-STOCHASTIC DIFFERENTIAL EQUATIONS

SVETLIN GEORGIEV

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Preface

This book is intended for readers who have had a course in iso-differential calculus and theory of probability. It can be used for a senior undergraduate course.

Chapter 1 represents a short introduction to the theory of iso-probability theory. They are defined iso-probability measure, iso-probability space, random iso-variable of the first, second, third, fourth and fifth kind, iso-expected values, iso-martingales, iso-Brownian motion, iso-Wiener processes, Paley-Wiener-Zygmund integral, Itô's iso-integral, and they are deducted some of their properties.

Chapter 2 is devoted on the iso-stochastic differential equations of the first, second and third kind, and for them they are proved the general existence and uniqueness theorems. They are given some methods for solving of some classes iso-stochastic differential equations.

Chapter 3 deals with the linear iso-stochastic differential equations.

The dependence on parameters and initial data is considered in Chapter 4.

In Chapter 5 is investigated the stability of the main classes iso-stochastic differential equations.

Iso-Stratonovich iso-integral and its properties are considered in Chapter 6.

I will be very grateful to anybody who wants to inform me about errors or just misprints, or wants to express criticism or other comments, to my e-mails svetlingeorgiev1@gmail.com, sgg2000bg@yahoo.com.

Svetlin Georgiev
Paris, France
August 15, 2014

CONTENTS

Chapter 1

A Crash Course in Basic Iso-Probability Theory

1.1 Basic Definitions

Let Ω be a nonempty set.

Definition 1.1.1. *A collection U of subsets of Ω with the properties*

(i) $\emptyset$, $\Omega \in U$,

(ii) *if* $A_1, A_2, \ldots \in U$, *then*

$$\bigcup_{k=1}^{\infty} A_k, \bigcap_{k=1}^{\infty} A_k \in U,$$

(iii) *if* $A \in U$, *then* $A^c := \Omega \backslash A \in U$,

will be called σ*-algebra.*

Below we will suppose that U is an σ-algebra of subsets of Ω. Let $\hat{T} : U \longrightarrow (0, \infty)$, $P : U \longrightarrow [0, \infty)$ be given. We note

$$\hat{P}(A) := \frac{P(A)}{\hat{T}(A)} \qquad \text{for} \qquad \forall A \in U.$$

Definition 1.1.2. *We call $\hat{P}$ an iso-probability measure provided*

(i) $\hat{P}(\emptyset) = 0$, $\hat{P}(\Omega) = \hat{I}(\Omega) := \frac{1}{\hat{T}(\Omega)}$,

(ii) *if* $A_1, A_2, \ldots \in U$, *then*

$$\hat{P}(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \hat{P}(A_k),$$

(iii) *if* $A_1, A_2, \ldots \in U$ *are disjoint sets, then*

$$\hat{P}(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \hat{P}(A_k).$$

Below we will suppose that $\hat{P}$ is an iso-probability measure on U.

Proposition 1.1.3. *For every $A \in U$ we have*

$$\hat{P}(A) \le \frac{1}{\hat{T}(\Omega)}.$$

Proof. Since

$$A \bigcup A^c = U, \qquad A \bigcap A^c = \emptyset,$$

we have

$$\tfrac{1}{\hat{T}(\Omega)} = \hat{P}(\Omega) = \hat{P}(A \bigcup A^c) = \hat{P}(A) + \hat{P}(A^c),$$

whereupon, because $\hat{P}(A) \ge 0$, $\hat{P}(A^c) \ge 0$, we conclude that

$$\hat{P}(A) \le \frac{1}{\hat{T}(\Omega)}.$$

□

Definition 1.1.4. *The triple $(\Omega, U, \hat{P})$ will be called an iso-probability space.*

Definition 1.1.5. **(i)** *A set $A \in U$ is called an event, points $\omega \in \Omega$ are sample paths.*

(ii) *$\hat{P}(A)$ is the iso-probability of the event A.*

(iii) *A property which is true except for an event of iso-probability zero is said to hold iso-almost surely(usually abbreviated "i-a.s.").*

Proposition 1.1.6. *Let A, $B \in U$, $A \subseteq B$,*

$$P(A) \le P(B), \qquad \hat{T}(A) \ge \hat{T}(B).$$

Then

$$\hat{P}(A) \le \hat{P}(B).$$

Proof. Since $\hat{T}(A) \ge \hat{T}(B)$, we have

$$\frac{1}{\hat{T}(A)} \le \frac{1}{\hat{T}(B)},$$

from here and from $P(A) \le P(B)$, we obtain

$$\hat{P}(A) = \frac{P(A)}{\hat{T}(A)} \le \frac{P(B)}{\hat{T}(B)} = \hat{P}(B).$$

□

Corollary 1.1.7. *If P is a probability measure on U and $\hat{T}(A) \ge \hat{T}(B)$, $A \subseteq B$, A, $B \in U$, then $\hat{P}(A) \le \hat{P}(B)$.*

Proposition 1.1.8. *Let A, $B \in U$, $A \subseteq B$, and*

$$P(A) \geq P(B), \qquad \hat{T}(A) \leq \hat{T}(B).$$

Then

$$\hat{P}(A) \geq \hat{P}(B).$$

Proof. Since $\hat{T}(A) \leq \hat{T}(B)$, then

$$\frac{1}{\hat{T}(A)} \geq \frac{1}{\hat{T}(B)}.$$

From here and from $P(A) \geq P(B)$, we get

$$\hat{P}(A) = \frac{P(A)}{\hat{T}(A)} \geq \frac{P(B)}{\hat{T}(B)} = \hat{P}(B).$$

□

Proposition 1.1.9. *Let $A \subset \Omega$. Then*

$$\hat{P}(A) + \hat{P}(A^c) = \frac{1}{\hat{T}(\Omega)}.$$

Proof. We note that

$$A \bigcup A^c = \Omega.$$

Then

$$\hat{P}(A \bigcup A^c) = \hat{P}(\Omega) = \frac{1}{\hat{T}(\Omega)}.$$

From here, since $A \bigcap A^c = \emptyset$, we get

$$\hat{P}(A \bigcup A^c) = \hat{P}(A) + \hat{P}(A^c)$$

and

$$\hat{P}(A) + \hat{P}(A^c) = \frac{1}{\hat{T}(\Omega)}.$$

□

Definition 1.1.10. *Let A, $B \subset \Omega$. If $\hat{P}(A) > 0$, then the quotient*

$$\hat{P}_A(B) = \frac{\hat{P}(A \bigcap B)}{\hat{P}(A)}$$

is defined to be conditional iso-probability of the event B under the condition A.

We can rewrite the conditional iso-probability as follows

$$\hat{P}_A(B) = \frac{\hat{P}(A\bigcap B)}{\hat{P}(A)}$$

$$= \frac{\frac{P(A\bigcap B)}{\hat{T}(A\bigcap B)}}{\frac{P(A)}{\hat{T}(A)}}$$

$$= \frac{P(A\bigcap B)\hat{T}(A)}{P(A)\hat{T}(A\bigcap B)}.$$

From the definition for conditional iso-probability, it follows

$$\hat{P}(A\bigcap B) = \hat{P}(A)\hat{P}_A(B).$$

Theorem 1.1.11. *(multiplicative theorem) For every $n \in \mathbb{N}$, $n \geq 2$, if $A_1, A_2, \ldots, A_n \in \Omega$, we have*

$$\hat{P}(A_1\bigcap A_2\bigcap \ldots \bigcap A_n) = \hat{P}(A_1)\hat{P}_{A_1}(A_2)\hat{P}_{A_1\bigcap A_2}(A_3)\bigcap \ldots \bigcap \hat{P}_{A_1\bigcap A_2\bigcap \ldots \bigcap A_{n-1}}(A_n).$$

Proof. The case $n = 2$ follows from the definition for conditional iso-probability. Now we consider the case $n = 3$. We have

$$\hat{P}(A_1\bigcap A_2\bigcap A_3) = \hat{P}(A_1\bigcap(A_2\bigcap A_3))$$

$$= \hat{P}(A_1)\hat{P}_{A_1}(A_2\bigcap A_3)$$

now we apply the case $n = 2$

$$= \hat{P}(A_1)\hat{P}_{A_1}(A_2)\hat{P}_{A_1\bigcap A_2}(A_3).$$

We suppose that the assertion is valid for some $n \in \mathbb{N}$, $n \geq 2$. We will prove the assertion in the case $n+1$.

Indeed,

$$\hat{P}(A_1\bigcap A_2\bigcap \ldots \bigcap A_n\bigcap A_{n+1}) = \hat{P}(A_1\bigcap(A_2\bigcap A_3\bigcap \ldots \bigcap A_{n+1}))$$

$$= \hat{P}(A_1)\hat{P}_{A_1}(A_2\bigcap A_3\bigcap \ldots \bigcap A_n\bigcap A_{n+1})$$

$$= \hat{P}(A_1)\hat{P}_{A_1}(A_2)\hat{P}_{A_1\bigcap A_2}(A_3\bigcap \ldots \bigcap A_n\bigcap A_{n+1})$$

$$\ldots$$

$$= \hat{P}(A_1)\hat{P}_{A_1}(A_2)\hat{P}_{A_1\bigcap A_2}(A_3)\ldots \hat{P}_{A_1\bigcap A_2\bigcap \ldots \bigcap A_n}(A_{n+1}).$$

□

Proposition 1.1.12. *Let $A \subset \Omega$. Then*

$$\hat{P}_A(\Omega) = 1.$$

Proof. From the definition for conditional iso-probability, we have

$$\hat{P}_A(\Omega) = \frac{\hat{P}(A\bigcap\Omega)}{\hat{P}(A)} = \frac{\hat{P}(A)}{\hat{P}(A)} = 1.$$

□

Proposition 1.1.13. *Let A, $B \subset \Omega$. Then*

$$\hat{P}_A(B) \geq 0.$$

Proof. We have that

$$\hat{P}_A(B) = \frac{\hat{P}(A\bigcap B)}{\hat{P}(A)},$$

from here, since $\hat{P}(A) > 0$ and $\hat{P}(A\bigcap B) \geq 0$, we conclude that $\hat{P}_A(B) \geq 0$. □

Proposition 1.1.14. *Let A, B, $C \subset \Omega$, $B\bigcap C = \emptyset$. Then*

$$\hat{P}_A(B\bigcup C) = \hat{P}_A(B) + \hat{P}_A(C).$$

Proof. We have

$$\hat{P}_A(B\bigcup C) = \frac{\hat{P}(A\bigcap(B\bigcup C))}{\hat{P}(A)}$$

$$= \frac{\hat{P}((A\bigcap B)\bigcup(A\bigcap C))}{\hat{P}(A)}$$

$$= \frac{\hat{P}(A\bigcap B)+\hat{P}(A\bigcap C)}{\hat{P}(A)}$$

$$= \hat{P}_A(B) + \hat{P}_A(C).$$

□

Theorem 1.1.15. *Let A_1, A_2, …, A_n are disjoint sets in Ω and*

$$\Omega = A_1\bigcup A_2\bigcup\ldots\bigcup A_n.$$

Let also, $X \subset \Omega$ be arbitrarily chosen. Then

$$\hat{P}(X) = \hat{P}(A_1)\hat{P}_{A_1}(X) + \cdots + \hat{P}(A_n)\hat{P}_{A_n}(X).$$

Proof. We have that

$$\Omega = A_1\bigcup A_2\bigcup\ldots\bigcup A_n.$$

Since

$$X = X\bigcap\Omega,$$

we get

$$X = X\bigcap\Omega = X\bigcap(A_1\bigcup A_2\bigcup\ldots\bigcup A_n)$$

$$= (X\bigcap A_1)\bigcup(X\bigcap A_2)\bigcup\ldots\bigcup(X\bigcap A_n).$$

Then

$$\hat{P}(X) = \hat{P}((X\bigcap A_1)\bigcup(X\bigcap A_2)\bigcup\ldots\bigcup(X\bigcap A_n))$$

$$= \hat{P}(X\bigcap A_1) + \hat{P}(X\bigcap A_2) + \cdots + \hat{P}(X\bigcap A_n)$$

$$= \hat{P}(A_1)\hat{P}_{A_1}(X) + \hat{P}(A_2)\hat{P}_{A_2}(X) + \cdots + \hat{P}(A_n)\hat{P}_{A_n}(X).$$

□

Proposition 1.1.16. *Let* A, $B \subset \Omega$, $\hat{P}(A) > 0$, $\hat{P}(B) > 0$. *Then*

$$\hat{P}_B(A) = \frac{\hat{P}(A)\hat{P}_A(B)}{\hat{P}(B)}.$$

Proof. We have

$$\hat{P}_A(B) = \frac{\hat{P}(A\bigcap B)}{\hat{P}(A)},$$

whereupon

$$\hat{P}(A\bigcap B) = \hat{P}_A(B)\hat{P}(A).$$

From here,

$$\hat{P}_B(A) = \frac{\hat{P}(A\bigcap B)}{\hat{P}(B)} = \frac{\hat{P}(A)\hat{P}_A(B)}{\hat{P}(B)}.$$

□

Definition 1.1.17. *The events* A_1, A_2, ..., A_m *will be called mutually independent if*

$$\hat{P}(A_1\bigcap A_2\bigcap\ldots\bigcap A_m) = \hat{T}(\Omega)\hat{P}(A_1)\hat{P}(A_2)\ldots\hat{P}(A_m).$$

Proposition 1.1.18. *Let* A_1, $A_2 \subset \Omega$ *be mutually independent. Then*

$$\hat{P}(A_1\bigcap A_2^c) = \hat{T}(\Omega)\hat{P}(A_1)\hat{P}(A_2^c).$$

Proof. Firstly, we will note

$$\hat{P}(A_1) = \hat{P}(A_1\bigcap\Omega)$$

$$= \hat{P}(A_1\bigcap(A_2\bigcup A_2^c))$$

$$= \hat{P}((A_1\bigcap A_2)\bigcup(A_1\bigcap A_2^c))$$

$$= \hat{P}(A_1\bigcap A_2) + \hat{P}(A_1\bigcap A_2^c),$$

i.e.,

$$\hat{P}(A_1\bigcap A_2^c) = \hat{P}(A_1) - \hat{P}(A_1\bigcap A_2),$$

whereupon

$$\hat{P}(A_1 \bigcap A_2^c) = \hat{P}(A_1) - \hat{T}(\Omega)\hat{P}(A_1)\hat{P}(A_2)$$

$$= \hat{T}(\Omega)\hat{P}(A_1)\left(\tfrac{1}{\hat{T}(\Omega)} - \hat{P}(A_2)\right)$$

$$= \hat{T}(\Omega)\hat{P}(A_1)\hat{P}(\Omega \backslash A_2)$$

$$= \hat{T}(\Omega)\hat{P}(A_1)\hat{P}(A_2^c).$$

□

As in above, one can prove the following assertion.

Proposition 1.1.19. *Let A_1, $A_2 \subset \Omega$ be mutually independent. Then*

$$\hat{P}(A_1^c \bigcap A_2) = \hat{T}(\Omega)\hat{P}(A_1^c)\hat{P}(A_2).$$

Proposition 1.1.20. *Let A_1, $A_2 \subset \Omega$ be mutually independent. Then*

$$\hat{P}(A_1^c \bigcap A_2^c) = \hat{T}(\Omega)\hat{P}(A_1^c)\hat{P}(A_2^c).$$

Proof. We have

$$\hat{P}(A_1^c) = \hat{P}(A_1^c \bigcap \Omega)$$

$$= \hat{P}(A_1^c \bigcap (A_2^c \bigcup A_2))$$

$$= \hat{P}((A_1^c \bigcap A_2^c) \bigcup (A_1^c \bigcap A_2))$$

$$= \hat{P}(A_1^c \bigcap A_2^c) + \hat{P}(A_1^c \bigcap A_2)$$

$$= \hat{P}(A_1^c \bigcap A_2) + \hat{T}(\Omega)\hat{P}(A_1^c)\hat{P}(A_2),$$

whereupon

$$\hat{T}(\Omega)\hat{P}(A_1^c)\left(\frac{1}{\hat{T}(\Omega)} - \hat{P}(A_2)\right) = \hat{P}(A_1^c \bigcap A_2^c),$$

or

$$\hat{P}(A_1^c \bigcap A_2^c) = \hat{T}(\Omega)\hat{P}(A_1^c)\hat{P}(A_2^c).$$

□

Definition 1.1.21. *Let A_1, A_2, ..., A_n, ... be events. Then the event*

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \left\{ \omega \in \Omega : \omega \text{ belongs to infinitely many of the } A_n \right\}$$

is called A_n infinitely often, abbreviated A_ni.o., i.e.,

$$A_n\text{i.o.} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

Lemma 1.1.22. *(iso-Borel-Cantelli lemma) Let A_1, A_2, ..., A_n, ..., be events such that $\hat{T}(A_n) \geq Q$ for every $n \in \mathbb{N}$, where Q is a positive constant. Let also, $P(A) \leq P(B)$ and $\hat{T}(A) \geq \hat{T}(B)$ for every $A, B \in U$, $A \subset B$. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then*

$$\hat{P}(A_n \text{i.o.}) = 0.$$

Proof. By definition we have

$$A_n \text{i.o.} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

Therefore

$$A_n \text{i.o.} \subset \bigcup_{m=n}^{\infty} A_m \qquad \text{for} \qquad \forall n \in \mathbb{N}.$$

Hence,

$$P(A_n \text{i.o.}) \leq P\left(\bigcup_{m=n}^{\infty} A_m\right),$$

$$\hat{T}(A_n \text{i.o.}) \geq \hat{T}\left(\bigcup_{m=n}^{\infty} A_m\right),$$

$$\frac{1}{\hat{T}(A_n \text{i.o.})} \leq \frac{1}{\hat{T}\left(\bigcup_{m=n}^{\infty} A_m\right)},$$

and

$$\hat{P}(A_n \text{i.o.}) = \frac{P(A_n \text{i.o.})}{\hat{T}(A_n \text{i.o.})}$$

$$\leq \frac{P\left(\bigcup_{m=n}^{\infty} A_m\right)}{\hat{T}\left(\bigcup_{m=n}^{\infty} A_m\right)}$$

$$= \hat{P}\left(\bigcup_{m=n}^{\infty} A_m\right)$$

$$\leq \sum_{m=n}^{\infty} \hat{P}(A_m)$$

$$= \sum_{m=n}^{\infty} \frac{P(A_m)}{\hat{T}(A_m)}$$

$$\leq \frac{1}{Q} \sum_{m=n}^{\infty} P(A_m)$$

$$\longrightarrow_{n \longrightarrow \infty},$$

because $\sum_{n=1}^{\infty} P(A_n) < \infty$. □

Definition 1.1.23. *A sequence $\{\hat{X}_k\}_{k=1}^{\infty}$ of random iso-variables converges in iso-probability to an iso-random variable $\hat{X}$, provided*

$$\lim_{k \longrightarrow \infty} \hat{P}(|\hat{X}_k - \hat{X}|) > \varepsilon$$

for every $\varepsilon > 0$.