

Volume 5

Foundations of Iso-Differential Calculus

Ordinary Iso-Differential Equations

Mathematics
Research
Developments

$$0 \leq \lim_{n \rightarrow +\infty} \frac{|a_n|}{\hat{T}_n} \leq \lim_{n \rightarrow +\infty} \frac{1}{p} = 0.$$

Svetlin Georgiev

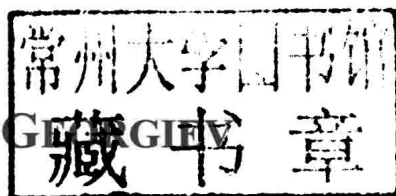
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MATHEMATICS RESEARCH DEVELOPMENTS

FOUNDATIONS OF ISO-DIFFERENTIAL CALCULUS

VOLUME 5 ISO-STOCHASTIC DIFFERENTIAL EQUATIONS

SVETLIN



GEORGIEV

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MATHEMATICS RESEARCH DEVELOPMENTS

**FOUNDATIONS OF ISO-DIFFERENTIAL
CALCULUS**

**VOLUME 5
ISO-STOCHASTIC DIFFERENTIAL
EQUATIONS**

MATHEMATICS RESEARCH DEVELOPMENTS

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Preface

This book is intended for readers who have had a course in iso-differential calculus and theory of probability. It can be used for a senior undergraduate course.

Chapter 1 represents a short introduction to the theory of iso-probability theory. They are defined iso-probability measure, iso-probability space, random iso-variable of the first, second, third, fourth and fifth kind, iso-expected values, iso-martingales, iso-Brownian motion, iso-Wiener processes, Paley-Wiener-Zygmund integral, Itô's iso-integral, and they are deduced some of their properties.

Chapter 2 is devoted on the iso-stochastic differential equations of the first, second and third kind, and for them they are proved the general existence and uniqueness theorems. They are given some methods for solving of some classes iso-stochastic differential equations.

Chapter 3 deals with the linear iso-stochastic differential equations.

The dependence on parameters and initial data is considered in Chapter 4.

In Chapter 5 is investigated the stability of the main classes iso-stochastic differential equations.

Iso-Stratonovich iso-integral and its properties are considered in Chapter 6.

I will be very grateful to anybody who wants to inform me about errors or just misprints, or wants to express criticism or other comments, to my e-mails svetlingeorgiev1@gmail.com, sgg2000bg@yahoo.com.

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Paris, France
August 15, 2014

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Chapter 1

A Crash Course in Basic Iso-Probability Theory

1.1 Basic Definitions

Let Ω be a nonempty set.

Definition 1.1.1. A collection U of subsets of Ω with the properties

(i) $\emptyset, \Omega \in U$,

(ii) if $A_1, A_2, \dots \in U$, then

$$\bigcup_{k=1}^{\infty} A_k, \bigcap_{k=1}^{\infty} A_k \in U,$$

(iii) if $A \in U$, then $A^c := \Omega \setminus A \in U$,

will be called σ -algebra.

Below we will suppose that U is an σ -algebra of subsets of Ω .

Let $\hat{T} : U \longrightarrow (0, \infty)$, $P : U \longrightarrow [0, \infty)$ be given. We note

$$\hat{P}(A) := \frac{P(A)}{\hat{T}(A)} \quad \text{for} \quad \forall A \in U.$$

Definition 1.1.2. We call \hat{P} an iso-probability measure provided

(i) $\hat{P}(\emptyset) = 0$, $\hat{P}(\Omega) = \hat{T}(\Omega) := \frac{1}{\hat{T}(\Omega)}$,

(ii) if $A_1, A_2, \dots \in U$, then

$$\hat{P}\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \hat{P}(A_k),$$

(iii) if $A_1, A_2, \dots \in U$ are disjoint sets, then

$$\hat{P}\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \hat{P}(A_k).$$

Below we will suppose that \hat{P} is an iso-probability measure on U .

Proposition 1.1.3. *For every $A \in U$ we have*

$$\hat{P}(A) \leq \frac{1}{\hat{T}(\Omega)}.$$

Proof. Since

$$A \cup A^c = U, \quad A \cap A^c = \emptyset,$$

we have

$$\frac{1}{\hat{T}(\Omega)} = \hat{P}(\Omega) = \hat{P}(A \cup A^c) = \hat{P}(A) + \hat{P}(A^c),$$

whereupon, because $\hat{P}(A) \geq 0$, $\hat{P}(A^c) \geq 0$, we conclude that

$$\hat{P}(A) \leq \frac{1}{\hat{T}(\Omega)}.$$

□

Definition 1.1.4. *The triple (Ω, U, \hat{P}) will be called an iso-probability space.*

Definition 1.1.5. (i) *A set $A \in U$ is called an event, points $\omega \in \Omega$ are sample paths.*

(ii) *$\hat{P}(A)$ is the iso-probability of the event A .*

(iii) *A property which is true except for an event of iso-probability zero is said to hold iso-almost surely (usually abbreviated "i-a.s.").*

Proposition 1.1.6. *Let $A, B \in U$, $A \subseteq B$,*

$$P(A) \leq P(B), \quad \hat{T}(A) \geq \hat{T}(B).$$

Then

$$\hat{P}(A) \leq \hat{P}(B).$$

Proof. Since $\hat{T}(A) \geq \hat{T}(B)$, we have

$$\frac{1}{\hat{T}(A)} \leq \frac{1}{\hat{T}(B)},$$

from here and from $P(A) \leq P(B)$, we obtain

$$\hat{P}(A) = \frac{P(A)}{\hat{T}(A)} \leq \frac{P(B)}{\hat{T}(B)} = \hat{P}(B).$$

□

Corollary 1.1.7. *If P is a probability measure on U and $\hat{T}(A) \geq \hat{T}(B)$, $A \subseteq B$, $A, B \in U$, then $\hat{P}(A) \leq \hat{P}(B)$.*

Proposition 1.1.8. *Let $A, B \in U$, $A \subseteq B$, and*

$$P(A) \geq P(B), \quad \hat{T}(A) \leq \hat{T}(B).$$

Then

$$\hat{P}(A) \geq \hat{P}(B).$$

Proof. Since $\hat{T}(A) \leq \hat{T}(B)$, then

$$\frac{1}{\hat{T}(A)} \geq \frac{1}{\hat{T}(B)}.$$

From here and from $P(A) \geq P(B)$, we get

$$\hat{P}(A) = \frac{P(A)}{\hat{T}(A)} \geq \frac{P(B)}{\hat{T}(B)} = \hat{P}(B).$$

□

Proposition 1.1.9. *Let $A \subset \Omega$. Then*

$$\hat{P}(A) + \hat{P}(A^c) = \frac{1}{\hat{T}(\Omega)}.$$

Proof. We note that

$$A \cup A^c = \Omega.$$

Then

$$\hat{P}(A \cup A^c) = \hat{P}(\Omega) = \frac{1}{\hat{T}(\Omega)}.$$

From here, since $A \cap A^c = \emptyset$, we get

$$\hat{P}(A \cup A^c) = \hat{P}(A) + \hat{P}(A^c)$$

and

$$\hat{P}(A) + \hat{P}(A^c) = \frac{1}{\hat{T}(\Omega)}.$$

□

Definition 1.1.10. *Let $A, B \subset \Omega$. If $\hat{P}(A) > 0$, then the quotient*

$$\hat{P}_A(B) = \frac{\hat{P}(A \cap B)}{\hat{P}(A)}$$

is defined to be conditional iso-probability of the event B under the condition A .

We can rewrite the conditional iso-probability as follows

$$\begin{aligned}\hat{P}_A(B) &= \frac{\hat{P}(A \cap B)}{\hat{P}(A)} \\ &= \frac{\frac{P(A \cap B)}{\hat{T}(A \cap B)}}{\frac{P(A)}{\hat{T}(A)}} \\ &= \frac{P(A \cap B) \hat{T}(A)}{P(A) \hat{T}(A \cap B)}.\end{aligned}$$

From the definition for conditional iso-probability, it follows

$$\hat{P}(A \cap B) = \hat{P}(A) \hat{P}_A(B).$$

Theorem 1.1.11. (*multiplicative theorem*) For every $n \in \mathbb{N}$, $n \geq 2$, if $A_1, A_2, \dots, A_n \in \Omega$, we have

$$\hat{P}(A_1 \cap A_2 \cap \dots \cap A_n) = \hat{P}(A_1) \hat{P}_{A_1}(A_2) \hat{P}_{A_1 \cap A_2}(A_3) \cap \dots \cap \hat{P}_{A_1 \cap A_2 \cap \dots \cap A_{n-1}}(A_n).$$

Proof. The case $n = 2$ follows from the definition for conditional iso-probability. Now we consider the case $n = 3$. We have

$$\begin{aligned}\hat{P}(A_1 \cap A_2 \cap A_3) &= \hat{P}(A_1 \cap (A_2 \cap A_3)) \\ &= \hat{P}(A_1) \hat{P}_{A_1}(A_2 \cap A_3) \\ \text{now we apply the case } n = 2 \\ &= \hat{P}(A_1) \hat{P}_{A_1}(A_2) \hat{P}_{A_1 \cap A_2}(A_3).\end{aligned}$$

We suppose that the assertion is valid for some $n \in \mathbb{N}$, $n \geq 2$. We will prove the assertion in the case $n + 1$.

Indeed,

$$\begin{aligned}\hat{P}(A_1 \cap A_2 \cap \dots \cap A_n \cap A_{n+1}) &= \hat{P}(A_1 \cap (A_2 \cap A_3 \cap \dots \cap A_n \cap A_{n+1})) \\ &= \hat{P}(A_1) \hat{P}_{A_1}(A_2 \cap A_3 \cap \dots \cap A_n \cap A_{n+1}) \\ &= \hat{P}(A_1) \hat{P}_{A_1}(A_2) \hat{P}_{A_1 \cap A_2}(A_3 \cap \dots \cap A_n \cap A_{n+1}) \\ &\dots \\ &= \hat{P}(A_1) \hat{P}_{A_1}(A_2) \hat{P}_{A_1 \cap A_2}(A_3) \dots \hat{P}_{A_1 \cap A_2 \cap \dots \cap A_n}(A_{n+1}).\end{aligned}$$

□

Proposition 1.1.12. Let $A \subset \Omega$. Then

$$\hat{P}_A(\Omega) = 1.$$

Proof. From the definition for conditional iso-probability, we have

$$\hat{P}_A(\Omega) = \frac{\hat{P}(A \cap \Omega)}{\hat{P}(A)} = \frac{\hat{P}(A)}{\hat{P}(A)} = 1.$$

□

Proposition 1.1.13. *Let $A, B \subset \Omega$. Then*

$$\hat{P}_A(B) \geq 0.$$

Proof. We have that

$$\hat{P}_A(B) = \frac{\hat{P}(A \cap B)}{\hat{P}(A)},$$

from here, since $\hat{P}(A) > 0$ and $\hat{P}(A \cap B) \geq 0$, we conclude that $\hat{P}_A(B) \geq 0$.

□

Proposition 1.1.14. *Let $A, B, C \subset \Omega$, $B \cap C = \emptyset$. Then*

$$\hat{P}_A(B \cup C) = \hat{P}_A(B) + \hat{P}_A(C).$$

Proof. We have

$$\begin{aligned} \hat{P}_A(B \cup C) &= \frac{\hat{P}(A \cap (B \cup C))}{\hat{P}(A)} \\ &= \frac{\hat{P}((A \cap B) \cup (A \cap C))}{\hat{P}(A)} \\ &= \frac{\hat{P}(A \cap B) + \hat{P}(A \cap C)}{\hat{P}(A)} \\ &= \hat{P}_A(B) + \hat{P}_A(C). \end{aligned}$$

□

Theorem 1.1.15. *Let A_1, A_2, \dots, A_n are disjoint sets in Ω and*

$$\Omega = A_1 \cup A_2 \cup \dots \cup A_n.$$

Let also, $X \subset \Omega$ be arbitrarily chosen. Then

$$\hat{P}(X) = \hat{P}(A_1)\hat{P}_{A_1}(X) + \dots + \hat{P}(A_n)\hat{P}_{A_n}(X).$$

Proof. We have that

$$\Omega = A_1 \cup A_2 \cup \dots \cup A_n.$$

Since

$$X = X \cap \Omega,$$

we get

$$\begin{aligned} X &= X \cap \Omega = X \cap (A_1 \cup A_2 \cup \dots \cup A_n) \\ &= (X \cap A_1) \cup (X \cap A_2) \cup \dots \cup (X \cap A_n). \end{aligned}$$

Then

$$\begin{aligned}
 \hat{P}(X) &= \hat{P}((X \cap A_1) \cup (X \cap A_2) \cup \dots \cup (X \cap A_n)) \\
 &= \hat{P}(X \cap A_1) + \hat{P}(X \cap A_2) + \dots + \hat{P}(X \cap A_n) \\
 &= \hat{P}(A_1)\hat{P}_{A_1}(X) + \hat{P}(A_2)\hat{P}_{A_2}(X) + \dots + \hat{P}(A_n)\hat{P}_{A_n}(X).
 \end{aligned}$$

□

Proposition 1.1.16. *Let $A, B \subset \Omega$, $\hat{P}(A) > 0$, $\hat{P}(B) > 0$. Then*

$$\hat{P}_B(A) = \frac{\hat{P}(A)\hat{P}_A(B)}{\hat{P}(B)}.$$

Proof. We have

$$\hat{P}_A(B) = \frac{\hat{P}(A \cap B)}{\hat{P}(A)},$$

whereupon

$$\hat{P}(A \cap B) = \hat{P}_A(B)\hat{P}(A).$$

From here,

$$\hat{P}_B(A) = \frac{\hat{P}(A \cap B)}{\hat{P}(B)} = \frac{\hat{P}(A)\hat{P}_A(B)}{\hat{P}(B)}.$$

□

Definition 1.1.17. *The events A_1, A_2, \dots, A_m will be called mutually independent if*

$$\hat{P}(A_1 \cap A_2 \cap \dots \cap A_m) = \hat{T}(\Omega)\hat{P}(A_1)\hat{P}(A_2) \dots \hat{P}(A_m).$$

Proposition 1.1.18. *Let $A_1, A_2 \subset \Omega$ be mutually independent. Then*

$$\hat{P}(A_1 \cap A_2^c) = \hat{T}(\Omega)\hat{P}(A_1)\hat{P}(A_2^c).$$

Proof. Firstly, we will note

$$\begin{aligned}
 \hat{P}(A_1) &= \hat{P}(A_1 \cap \Omega) \\
 &= \hat{P}(A_1 \cap (A_2 \cup A_2^c)) \\
 &= \hat{P}((A_1 \cap A_2) \cup (A_1 \cap A_2^c)) \\
 &= \hat{P}(A_1 \cap A_2) + \hat{P}(A_1 \cap A_2^c),
 \end{aligned}$$

i.e.,

$$\hat{P}(A_1 \cap A_2^c) = \hat{P}(A_1) - \hat{P}(A_1 \cap A_2),$$

whereupon

$$\begin{aligned}
 \hat{P}(A_1 \cap A_2^c) &= \hat{P}(A_1) - \hat{T}(\Omega) \hat{P}(A_1) \hat{P}(A_2) \\
 &= \hat{T}(\Omega) \hat{P}(A_1) \left(\frac{1}{\hat{T}(\Omega)} - \hat{P}(A_2) \right) \\
 &= \hat{T}(\Omega) \hat{P}(A_1) \hat{P}(\Omega \setminus A_2) \\
 &= \hat{T}(\Omega) \hat{P}(A_1) \hat{P}(A_2^c).
 \end{aligned}$$

□

As in above, one can prove the following assertion.

Proposition 1.1.19. *Let $A_1, A_2 \subset \Omega$ be mutually independent. Then*

$$\hat{P}(A_1^c \cap A_2) = \hat{T}(\Omega) \hat{P}(A_1^c) \hat{P}(A_2).$$

Proposition 1.1.20. *Let $A_1, A_2 \subset \Omega$ be mutually independent. Then*

$$\hat{P}(A_1^c \cap A_2^c) = \hat{T}(\Omega) \hat{P}(A_1^c) \hat{P}(A_2^c).$$

Proof. We have

$$\begin{aligned}
 \hat{P}(A_1^c) &= \hat{P}(A_1^c \cap \Omega) \\
 &= \hat{P}(A_1^c \cap (A_2^c \cup A_2)) \\
 &= \hat{P}((A_1^c \cap A_2^c) \cup (A_1^c \cap A_2)) \\
 &= \hat{P}(A_1^c \cap A_2^c) + \hat{P}(A_1^c \cap A_2) \\
 &= \hat{P}(A_1^c \cap A_2) + \hat{T}(\Omega) \hat{P}(A_1^c) \hat{P}(A_2),
 \end{aligned}$$

whereupon

$$\hat{T}(\Omega) \hat{P}(A_1^c) \left(\frac{1}{\hat{T}(\Omega)} - \hat{P}(A_2) \right) = \hat{P}(A_1^c \cap A_2^c),$$

or

$$\hat{P}(A_1^c \cap A_2^c) = \hat{T}(\Omega) \hat{P}(A_1^c) \hat{P}(A_2^c).$$

□

Definition 1.1.21. *Let $A_1, A_2, \dots, A_n, \dots$ be events. Then the event*

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \left\{ \omega \in \Omega : \omega \text{ belongs to infinitely many of the } A_n \right\}$$

is called A_n infinitely often, abbreviated A_n i.o., i.e.,

$$A_n \text{ i.o.} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

Lemma 1.1.22. (*iso-Borel-Cantelli lemma*) Let $A_1, A_2, \dots, A_n, \dots$, be events such that $\hat{T}(A_n) \geq Q$ for every $n \in \mathbb{N}$, where Q is a positive constant. Let also, $P(A) \leq P(B)$ and $\hat{T}(A) \geq \hat{T}(B)$ for every $A, B \in U, A \subset B$. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then

$$\hat{P}(A_n \text{ i.o.}) = 0.$$

Proof. By definition we have

$$A_n \text{ i.o.} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

Therefore

$$A_n \text{ i.o.} \subset \bigcup_{m=n}^{\infty} A_m \quad \text{for} \quad \forall n \in \mathbb{N}.$$

Hence,

$$P(A_n \text{ i.o.}) \leq P(\bigcup_{m=n}^{\infty} A_m),$$

$$\hat{T}(A_n \text{ i.o.}) \geq \hat{T}(\bigcup_{m=n}^{\infty} A_m),$$

$$\frac{1}{\hat{T}(A_n \text{ i.o.})} \leq \frac{1}{\hat{T}(\bigcup_{m=n}^{\infty} A_m)},$$

and

$$\begin{aligned} \hat{P}(A_n \text{ i.o.}) &= \frac{P(A_n \text{ i.o.})}{\hat{T}(A_n \text{ i.o.})} \\ &\leq \frac{P(\bigcup_{m=n}^{\infty} A_m)}{\hat{T}(\bigcup_{m=n}^{\infty} A_m)} \\ &= \hat{P}(\bigcup_{m=n}^{\infty} A_m) \\ &\leq \sum_{m=n}^{\infty} \hat{P}(A_m) \\ &= \sum_{m=n}^{\infty} \frac{P(A_m)}{\hat{T}(A_m)} \\ &\leq \frac{1}{Q} \sum_{m=n}^{\infty} P(A_m) \\ &\longrightarrow_{n \rightarrow \infty} 0, \end{aligned}$$

because $\sum_{n=1}^{\infty} P(A_n) < \infty$. □

Definition 1.1.23. A sequence $\{\hat{X}_k\}_{k=1}^{\infty}$ of random iso-variables converges in iso-probability to an iso-random variable \hat{X} , provided

$$\lim_{k \rightarrow \infty} \hat{P}(|\hat{X}_k - \hat{X}|) > \varepsilon$$

for every $\varepsilon > 0$.