



# ANALYSIS AND ADJUSTMENT OF SURVEY MEASUREMENTS

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# Preface

This book was written in response to recommendations made at the Ninth National Surveying Teachers' Conference, held June 1977 in Fredericton, New Brunswick, Canada. The theme of the Conference was the evaluation of resource materials in surveying and adjustment. After reviewing existing material on the adjustment of survey data, a need was identified for a new book at the introductory level. This book is intended to be a first-level, introductory treatment of the subject to meet that need.

The authors are also aware that there are many practicing surveyors who are eager to gain basic knowledge about the analysis and adjustment of survey measurements, but have been unable to achieve this goal in the past because of the unsuitability of available literature. Therefore, this book has been written as much for the practicing surveyor as for the beginning student. Indeed, the way the topics are systematically developed in the book makes it very useful as a handy reference for those who currently perform survey analysis and adjustment.

The book has been organized in a manner which guides the reader gradually through very simple concepts and techniques to more sophisticated methods. Knowledge of the usual algebra, geometry and elementary calculus required for surveying is assumed. Matrix algebra, which is becoming more and more a standard tool for the surveyor, is slowly and methodically introduced and incorporated throughout the book. As an aid to the reader who has no prior knowledge of matrix algebra, Appendix A can be used for self-study. Alternatively, it can be used to introduce matrix algebra in a few classes, either in a regular course or in workshops for practitioners.

The book has ten chapters. Chapter 1 introduces some basic concepts relating to measurement, error, probability and reliability, and concludes with a brief treatment of significant figures. Chapter 2 introduces the reader to the propagation of known errors and the important process of linearization. Chapter 3 is devoted to

the concept of adjustment, beginning with simple adjustment procedures and then introducing the least squares method. No attempt is made at this point to introduce weights or even to give much derivation. Instead, the least squares adjustment principle is demonstrated through several worked examples, allowing the reader to grasp the concept of, and reason for, least squares adjustment without being encumbered by much mathematical analysis. It is in Chapter 4 that the reader is introduced to the mathematical derivation of two basic techniques of least squares (adjustment of indirect observations, and adjustment of observations only) and to the concept of weight. As is true throughout the book, Chapter 4 contains an abundance of worked numerical examples.

It is anticipated that after Chapter 4 the reader will be ready for an introduction to the elementary concepts in probability that are given in Chapter 5. Here again, derivations are kept to a minimum, while worked examples are numerous. The chapter covers such basic topics as random variables, probability distributions, expectation, variance and covariance, and concludes with an introduction to sampling. This prepares the reader for Chapter 6, which covers variance-covariance propagation. In addition to derivation of the propagation laws, this chapter also includes the application of variance-covariance propagation to the two least squares techniques introduced in Chapter 4. With knowledge of propagation, the reader can then appreciate pre-analysis of survey measurements addressed in Chapter 7. As this topic is quite extensive, only an introduction to it is given.

The material in the first seven chapters can be used as the basis for a relatively short and elementary course in 2-, 3- or 4-year technology programs that cannot devote more time to the subject. It can also be used for an intensive 1–2-week course or workshop organized for practicing surveyors. For professional programs in land surveying, engineering surveying or survey science and technology designed to cover more material, course coverage can be extended to include Chapters 8, 9 and 10.

Chapter 8 builds on preceding chapters and introduces well-established statistical concepts that can be used in survey analysis. These include sampling distributions and statistics, statistical estimation and testing, and error ellipses. In treating these topics, their use in surveying is emphasized. Chapter 9 presents the general approach to least squares adjustment and establishes its relation to the two basic techniques presented earlier in Chapter 4. Chapter 10 concludes the book with some applications related to plane coordinate surveys.

The authors wish to acknowledge the assistance of Dr. G. Warren Marks who discussed the contents of the book and continuously provided much encouragement for its completion. The assistance of colleagues at Purdue University and the

University of Toronto whose problems were adopted in the book is also much appreciated. Many thanks are also due to Mrs. Bonnie Stiff who so patiently and diligently typed the final manuscript, and to D. Schwartz, J. Thurgood, and F. Paderes who assisted with editing the galleys and page proofs. And to our wives, LaVerne and Ankie, may we express a very special note of gratitude for persevering with us, without complaint, throughout the entire period of the book's preparation.

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# 1

## Introduction

### 1.1. CONCEPTS OF MEASUREMENT AND ERROR

The surveyor may be called upon to perform a variety of tasks in connection with a survey project, from initial project design to final presentation of results in some specified form. Much of what the surveyor does, however, involves measurements, including their adjustment and analysis in the office as well as their acquisition in the field. If the surveyor is to acquire, adjust, and analyze measurements intelligently, he must understand what the process of measurement is.

Measurement must involve observation. No measurement is made until something is observed. Accordingly, the terms *measurement* and *observation* are often used synonymously.

Although we may think of a measurement as a single act, a typical survey measurement may involve several elementary operations, including such observing operations as centering, pointing, matching, setting, and reading. Yet at the end of all these operations a single numerical value is used to represent the “measurement” or “observation” of the quantity that we are seeking.

Let us consider, for example, the relatively simple task of using a 30 m steel tape suspended above the ground to measure the distance between two survey stations that are slightly less than 20 m apart. The following elementary operations are performed:

1. The head and rear tapemen take up positions next to their respective survey stations and hold the tape above the ground, approximately in line with the stations.
2. The rear tapeman suspends his plumb bob from the tape, setting and holding the string on the 20 m graduation.
3. The rear tapeman then centers his plumb bob over his survey station.
4. The head tapeman suspends his plumb bob from the tape, with the plumb bob string held near the zero graduation, and applies tension to

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the tape. (The rear tapeman, of course, must pull equally in the opposite direction while keeping his plumb bob centered over his station.)

5. While maintaining tension, the head tapeman shifts his plumb bob string along the tape until his plumb bob is centered over his survey station.
6. The head tapeman then reads the position of his string on the tape.
7. The measured distance is obtained by subtracting the tape reading in step 6 from the 20 m value set in step 2.

It is obvious that steps 2, 3, 5, and 6 are observing operations of setting, centering, centering, and reading, respectively, all of which are necessary to obtain a measurement of the distance between the two survey stations. This clearly demonstrates that even a simple measurement may involve several elementary operations, and that what is regarded as an observation of a desired quantity, such as the distance between two survey stations, is not just a single reading of a scale but the result of several steps in an observing process.

Even steps 1 through 7 are not the whole story in getting a reliable measurement of the distance. For some purposes the value obtained in step 7 may be satisfactory; for other purposes, it is not. If a "better" measurement is required, the step 7 value must be corrected for such things as length of tape, temperature, and sag. To evaluate the appropriate corrections, more things have to be observed, because the tape must be compared with a standard, the temperature must be observed on a thermometer, and the pull applied to the tape must be observed on a tension handle or similar device. For a precise survey measurement, these corrections and others are every bit as important as steps 1 through 7. A measurement is, indeed, the result of several operations, each of which makes some contribution to the measurement's ultimate usefulness.

Measurement is a process that is subject to variation. Variation can occur if some aspect of the measurement, such as temperature, is not taken into account. If, for example, several measurements of the distance between the two survey stations are made with the steel tape, and a change in temperature takes place while the measurements are being made, there will be a corresponding change in the length of the tape and consequently in the readings on the tape. If no correction is made for the effect of temperature, the resulting measurements will exhibit variation due to the temperature change. Variation is also a natural consequence of the elementary observing operations themselves. No observation can be repeated exactly, because of limitations in the instrumentation used and in the ability of the observer to center, point, match, set, and read. The small variations which occur in the ele-

mentary operations produce corresponding variations in the measurement.

Since all measurements are subject to variation, it follows that no quantity that is measured is completely determinable. We may seek a fixed value for a quantity that we conceive to be the *true value*, but what we get in reality is nothing more than an *estimate* of the true value. Mathematically, the measurement or observation must be looked upon as a variable. More will be said about this later.

It must be emphasized that the variation in the values obtained for a measured quantity is a natural phenomenon which is to be expected even when the conditions under which the measurements are made are essentially constant. If we are to expect variation, then we must expect a difference between a measured value of a quantity and its true value, whatever it may be. This difference is known as the *error* in the measured value. Although the English meaning of the term *error* may suggest that something wrong has been committed, errors in measurement should not be so regarded, unless perhaps they are what are called *gross errors* (see Section 1.2).

The study of observational errors and their behavior is essentially equivalent to the study of observations themselves. In other words, what has been classically referred to as the *theory of errors* is equivalent to what is now known as the *theory of observations*.

If  $\tau$  denotes the true value of a quantity (a distance, angle, etc.), and  $x$  is its observed value, then the error in  $x$  is defined as:

$$\epsilon = x - \tau \quad (1-1)$$

Since we will never really know what the value of  $\tau$  is, we will never know what the exact value of  $\epsilon$  is. However, if we are able to obtain by some means a good estimate of  $\tau$ , we can use this estimate in place of  $\tau$  as reference for expressing variation in the observed values. If  $\hat{x}$  denotes the estimate of  $\tau$ , then the difference between  $\hat{x}$  and the observed value,  $x$ , is defined as the residual,  $v$ ; specifically,

$$v = \hat{x} - x \quad (1-2)$$

The residual,  $v$ , is the quantity that is actually used to express variation in the measurement.

## 1.2. TYPES OF ERRORS

Errors have been traditionally classified into three types: (1) *gross errors*, (2) *systematic errors*, and (3) *random errors*. Each type will be discussed separately.

### Gross Errors

Gross errors are the results of blunders or mistakes that are due to carelessness of the observer. For example, the observer may make a pointing on the wrong survey target, or he may read a scale or dial incorrectly, or he may read the wrong scale, or record the wrong value of a reading by transposing numbers (e.g., recording 41.56 m as 41.65 m). There are any number of mistakes an observer can make if he is inattentive.

If a survey is to have any usefulness at all, mistakes and blunders cannot be tolerated. Good field procedures are designed to assist in detecting mistakes. These procedures include:

1. Careful checking of all pointings on survey targets.
2. Taking multiple readings on scales and checking for reasonable consistency.
3. Verifying recorded data by rereading scales.
4. Repeating entire measurements independently and checking for consistency.
5. Using simple geometric or algebraic checks, such as comparing the sum of three measured angles in a plane triangle with  $180^\circ$ .

It is very important to safeguard against the occurrence of mistakes. If they do occur, they must be detected and eliminated from the survey measurements before such measurements can be used.

### Systematic Errors

Systematic errors are so called because they occur according to some deterministic system which, when known, can be expressed by some functional relationship. If, for example, the expansion of a steel tape is essentially linear with respect to temperature, and the coefficient of thermal expansion is known, a functional relationship between the temperature and the expansion of the tape can be established. If the length of the tape at some specified standard temperature is taken as reference, the change in the length of the tape from this reference caused by change in temperature from its standard value is classified as a systematic error.

A systematic error follows a pattern which will be duplicated if the measurement is repeated under the same conditions. For example, measuring a distance with a steel tape that is too short will result in the same systematic error if the same tape is used by the same tapemen to measure the same distance under the same conditions of temperature, pull, support, and slope.

A systematic error is called a *constant* error if its magnitude and sign

remain the same throughout the measuring process. It is *counteracting* if its sign changes while its magnitude remains the same.

The *system* underlying a systematic error may depend on the observer, the instrument used, the physical or environmental conditions at the time the measurement is made, or any combination of these factors.

The personal bias of an observer leads to systematic errors that may be constant or counteracting, depending on the observation procedure. If the conditions of observation vary, the natural senses of vision and hearing of the observer may vary as well and his personal error becomes variable, too.

Imperfect instrument construction, or incomplete instrument adjustment can lead to instrumental errors that are systematic. Imperfect construction includes such things as variation in scale graduations and eccentricity in centering components. Incomplete instrument adjustment includes such things as not making the axis of collimation (telescope axis) of a theodolite perpendicular to the instrument's tilting (horizontal) axis.

Since survey measurements are acquired in the field, they are affected by many physical and environmental factors. Temperature, pull and terrain slope, for example, affect taped distances, while humidity and atmospheric pressure as well as temperature affect electro-optical distance measurements (EDM), angle measurements, and leveling. All of these effects are functionally expressible in terms of the factors which cause them and so are classified as systematic errors.

All sources of systematic error so far discussed are related directly to the observational operations. However, systematic errors can also occur through simplification of the geometry or mathematical model chosen to represent the survey. If, for example, a plane triangle instead of a spherical triangle is used to connect three survey stations that are spaced several kilometers apart, the spherical excess will emerge as a systematic error.

In the reduction of survey measurements, it is important to detect and correct for all possible systematic errors.

## Random Errors

After all blunders are detected and removed, and the measurements are corrected for all known systematic errors, there will still remain some variation in the measurements. This variation results from observational errors which have no known functional relationship based upon a deterministic system. These errors, instead, have random behavior, and must be treated accordingly.

It was stated earlier that a measurement or observation is looked upon mathematically as a variable. More specifically, it is a *random variable*



because it includes error components which exhibit random behavior. Indeed, the random errors themselves are random variables.

Whereas systematic variations are dealt with mathematically using functional relationships or models, random variables must use *probability models*. Some elementary concepts in probability theory are introduced in the following section. More will be discussed in Chapter 5.

### 1.3. ELEMENTARY CONCEPTS IN PROBABILITY

Let us assume that a distance is measured a very large number of times and that all measurements are free of gross errors and corrected for all systematic errors. Whatever variation remains in the measurements is caused by random errors only. Although it is not possible to correct measurement for specific random errors from knowledge of the measurement system, it is possible to study their collective behavior from their *frequency distribution*. It is the frequency distribution that is used as basis for constructing the probability model for the measurements.

#### EXAMPLE 1-1

A distance of about 810 m is measured 200 times. All measurements are free of gross errors and are corrected for systematic errors. The corrected values are expressed to 0.01 m. It is noted that after correcting for the systematic errors the resulting variation in the measurements ranges from 810.11 m to 810.23 m, distributed as follows:

VALUE OF MEASUREMENT (m)	NUMBER OF MEASUREMENTS
810.11	1
810.12	3
810.13	7
810.14	19
810.15	20
810.16	36
810.17	38
810.18	29
810.19	24
810.20	10
810.21	11
810.22	0
810.23	2

Evaluate and plot the relative frequencies of occurrence for all listed values.