Introductory Combinatorics

Richard A. Brualdi

Introductory Combinatorics

Richard A. Brualdi
Department of Mathematics
University of Wisconsin

NORTH-HOLLAND
NEW YORK OXFORD AMSTERDAM

ELSEVIER NORTH-HOLLAND, INC.
52 Vanderbilt Avenue, New York, New York 10017

NORTH-HOLLAND PUBLISHING COMPANY
P.O. Box 211
Amsterdam, The Netherlands

Library of Congress Cataloging in Publication Data
Brualdi, Richard A
Introductory combinatorics.
Bibliography: p.
Includes index.
1. Combinatorial analysis. I. Title.
QA164.B76 511'.6 76-30436
ISBN 0-7204-8610-6

Manufactured in the United States of America

Preface

This book is a direct outgrowth of a course called "Introduction to Combinatorics" that I have taught at the University of Wisconsin—Madison once a year for the past several years. With the growing importance of combinatorial methods in mathematics and in applications and its recognition as a legitimate mathematical discipline, I had felt that an introductory course in combinatorics was needed. The course was initiated with the Department of Mathematics but was later cross-listed by the Computer Sciences Department and the Statistics Department. Thus students at the University of Wisconsin—Madison have obtained university credit for it as either a mathematics, a computer sciences, or a statistics course. The prerequisite for the course is two semesters of calculus, although not much use is made of calculus (the notable exceptions occur in Chapter 7 on generating functions, where use is made of power series, and in Chapter 4 on binomial coefficients, where in a couple of instances some use of differentiation and integration is made in order to derive identities). The reason for the prerequisite was mainly to ensure that students had a moderate amount of mathematical sophistication; the reason it was no more than two semesters was that I hoped to attract some students in other disciplines in which the ideas and methods of combinatorics are becoming increasingly useful. Thus I have had students in the course who were studying subjects such as linguistics, biology, and electrical engineering. A number of students have been secondary education majors emphasizing mathematics. It seems to me that a course in combinatorics is excellent for prospective high-school mathematics teachers, because many combinatorial ideas are accessible early in one's mathematical development. A course in combinatorics is also, I believe, an ideal vehicle with which to show in a non-technical way the excitement, charm, and liveliness of mathematics as well as its usefulness in the outside world. Although the course I have taught at Wisconsin is intended as an undergraduate course, I have had some graduate students in computer sciences and statistics in my classes.

There is more material in the book than can be covered in a one-semester undergraduate course. The last time I taught the course at Madison, I covered Chapters 1 through 10 and part of 11 (up to the proof of the 5-color theorem for planar graphs), but in retrospect this was too much material for such a short time. A more reasonable one-semester course would be Chapters 1 through 7 (possibly omitting Chapter 7 if time is short) and then two of Chapters 8, 9, and 10. If there is additional time, then some topics can be selected from Chapters 11 and 12. For a more leisurely two-semester course, Chapters 1 through 8 can be covered in the first semester and Chapters 9 through 12 in the second semester. The interrelation of the chapters is as follows. Chapters 1 through 7 form a sequence and should be covered in order. Chapter 8 is a unit in itself, although the general problem treated in this chapter is introduced in Section 1.1. Likewise, Chapter 9 is a separate unit apart from its introduction in Section 1.5 and the use of Theorem 8.1.1 (from Chapter 8) in the proof of Theorem 9.3.3. Section 9.1 also contains a discussion of finite fields, including the construction of finite fields using polynomial rings. It is possible to limit one's discussion of finite fields to the field of integers modulo a prime p and to limit the application of finite fields in subsequent sections of Chapter 9 in the same way. Chapters 10 and 11 also form a separate unit and should be covered in order—again, there is discussion in Section 1.4 of one of the problems treated in Chapter 11 (the 4-color problem), and in Section 1.6 of one of the problems treated in Chapter 10 (the shortest-route problem). Chapter 12 is a little more difficult to classify. Sections 12.1 and 12.2 do not depend substantially on any of the previous material. Sections 12.3 and 12.4 form a unit and contain the marriage problem treated in Chapter 8 as a special case. Section 12.5 is also a separate unit and contains a generalization of the marriage problem.

The exercises—there are nearly 500 of them—should form an important part of a course given from this book. In order to obtain a thorough understanding of the material, the serious student should attempt a large number of problems. If one cannot apply the ideas and methods, then one has not progressed very far. A few of the exercises are starred; these seemed to the author to be at a much higher level than the others. Many of the exercises are routine; many are challenging but not overwhelming.

I wish to express my appreciation to my students who have tolerated an earlier version of this book and have kindly pointed out numerous errors. I am very grateful to Margaret Higbie for the encouragement she has given me during the preparation and production of this book. Also I would like to thank the staff of Elsevier North-Holland for the efficient and professional manner in which the book has been expedited. I am grateful to Herbert J. Ryser who first introduced me to the study of combinatorics.

RICHARD A. BRUALDI

February 1977 *Paris, France*

Contents

Chapter 1
What is Combinatorics?

It would be surprising indeed if a reader of this book had never solved a combinatorial problem. Have you ever counted the number of games n teams would play if each team played every other team exactly once? Have you ever constructed magic squares? Have you ever attempted to trace through a network without removing your pencil from the paper and without tracing any part of the network more than once? Have you ever counted the number of poker hands which are full houses in order to determine what the odds against a full house are? These are all combinatorial problems. As they might suggest, combinatorics has its historical roots in mathematical recreations and games. Many problems that were studied in the past either for amusement or for their aesthetic appeal are today of great importance in pure and applied science. Today combinatorics is an important branch of mathematics, and its influence is expanding rapidly. Part of the reason for the tremendous growth of combinatorics in the past decade has been the phenomenal impact that computers have had and continue to have in our society. Because of their lightning speed, computers have been able to solve large-scale problems that previously had been unthinkable. But computers do not function alone. They need to be programmed to perform. The basis for these programs often consists of combinatorial algorithms for the solutions of problems. Another reason for the growth of combinatorics is its applicability to disciplines that had previously had little serious contact with mathematics. Thus we find that the ideas and techniques of combinatorics are being used not only in the traditional area of mathematical application, namely the physical sciences, but also in the social and biological sciences.

Combinatorics is concerned with arrangements of the objects of a set into patterns. Two general types of problems occur repeatedly.

(i) *Existence of the arrangement*. If one wants to arrange the objects of a set so that certain conditions are fulfilled, it may be far from clear whether or not such an arrangement is possible. This needs to be determined. If the arrangement is not always possible, it is then appropriate to ask under what conditions, necessary and sufficient, the desired arrangement can be achieved.

(ii) *Enumeration or classification of the arrangements.* If a specified arrangement is possible, there may be several ways of achieving it. If so, one may want to count their number or to classify them into types.

Although both existence and enumeration can be considered for any combinatorial problem, it usually happens in practice that if the existence question requires extensive study, the enumeration problem is unmanageable. However, if the existence of a specified arrangement causes no difficulty, it may be possible to count the number of ways of achieving the arrangement. In exceptional cases (when their number is small), the arrangements can be listed. Thus many combinatorial problems are of the form "Is it possible to arrange...?" or "Does there exist a...?" or "In how many ways can...?" or "Count the number of...".

A third combinatorial problem that occurs in conjunction with (i) is

(iii) *Study of a known arrangement.* After one has done the (possibly difficult) work of constructing an arrangement satisfying certain specified conditions, its properties and structure can then be investigated. Such structure may have implications for the classification problem (ii) and also for potential applications.

More generally, *combinatorics is concerned with the analysis of discrete structures and relations*.

One of the principal tools of combinatorics for verifying discoveries is *mathematical induction*. Induction is often a powerful procedure, and it is especially so in combinatorics. It is often easier to prove a stronger result than a weaker result with mathematical induction. Although it is necessary to verify more in the inductive step, the inductive hypothesis is stronger.

But it is generally true that the solutions of combinatorial problems require *ad hoc* methods. One cannot in general fall back onto known results or axioms. One must study the situation, develop some insight, and use one's own ingenuity for the solution of the problem. I do not mean to imply that there are no general principles or methods that can be applied. The inclusion-exclusion principle, the so-called pigeonhole principle, and the methods of recurrence relations and generating functions are all examples of general principles and methods which we will take up in later chapters. But often to see that they can be applied and to apply them requires cleverness. Thus experience in solving combinatorial problems is very important.

In order to make the preceding discussion more meaningful, let us now turn to a few examples of combinatorial problems. They vary from relatively simple problems (but requiring ingenuity for solution) to problems whose solution was a major achievement in combinatorics.

1.1 EXAMPLE. PERFECT COVERS OF CHESSBOARDS

Consider an ordinary chessboard which is divided into 64 squares in 8 rows and 8 columns. Suppose there is available a supply of identical-shaped dominoes, pieces which cover exactly two adjacent squares of the chessboard. Is it possible to arrange 32 dominoes on the chessboard so that no 2 dominoes overlap, every domino covers 2 squares, and all the squares of the chessboard are covered? We call such an arrangement a *perfect cover* of the chessboard by dominoes. This is an easy arrangement problem, and one quickly can construct many different perfect covers. It is difficult but nonetheless possible to enumerate the number of different perfect covers. This was found by M. E. Fischer[1] in 1961 to be $12{,}988{,}816 = 2^4 \times (901)^2$. The ordinary chessboard can be replaced by a more general chessboard divided into mn squares lying in m rows and n columns. A perfect cover need not exist now. Indeed, there is no perfect cover for the 3-by-3 board. For which values of m and n does the m-by-n chessboard have a perfect cover? It is not difficult to see that an m-by-n chessboard will have a perfect cover if and only if at least one of m and n is even, or equivalently, if and only if the number of squares of the chessboard is even. Fischer has derived general formulae involving trigonometric functions for the number of different perfect covers for the m-by-n chessboard. This problem is equivalent to a famous problem in molecular physics known as the *dimer problem*. It originated in the investigation of the absorption of diatomic molecules (dimers) on surfaces. The squares of the chessboard correspond to molecules, while the dominoes correspond to the dimers.

Consider once again the 8-by-8 chessboard, and with a pair of scissors cut out two diagonally opposite corner squares. Is it possible to arrange 31 dominoes to obtain a perfect cover of this "pruned" board? Although the pruned board is very close to being the 8-by-8 chessboard, which has over twelve million perfect covers, it has no perfect cover. The proof of this is an example of simple but clever combinatorial reasoning. In an ordinary 8-by-8 chessboard the squares are alternately colored black and white, there being 32 black and 32 white squares. If we cut out 2 diagonally opposite corner squares, we have removed 2 squares of the same color, say white. This leaves 32 black and 30 white squares. But each domino covers

[1]Statistical Mechanics of Dimers on a Plane Lattice, *Physical Review* **124** (1961), 1664–1672.

1 black and 1 white square, so that 31 dominoes on the board must cover 31 black and 31 white squares. Therefore the pruned board can have no perfect cover.

More generally, one can take an *m*-by-*n* chessboard whose squares are alternately black and white and arbitrarily cut out some squares, leaving a pruned board. When does a pruned board have a perfect cover? Using the reasoning above, we conclude that the pruned board must have an equal number of black and white squares for a perfect cover to exist. But this is not sufficient, as the example in Figure 1.1 indicates.

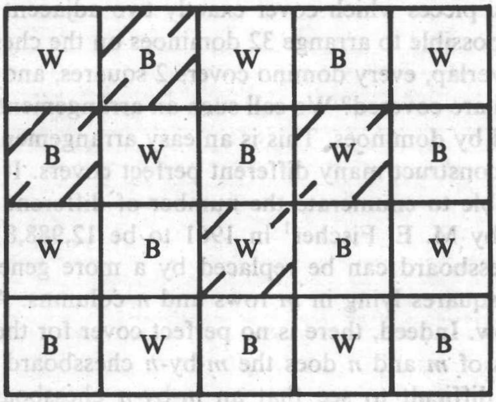

Figure 1.1

Thus we ask: What are necessary and sufficient conditions for a pruned board to have a perfect cover? We will return to this problem in Chapter 8 and obtain a complete solution using the theory of systems of distinct representatives. There a practical formulation of this problem in terms of assigning qualified applicants to jobs will be given.

1.2 EXAMPLE. CUTTING A CUBE

Consider a block of wood in the shape of a cube, 3 feet on an edge. It is desired to cut the cube into 27 smaller cubes, 1 foot on an edge. What is the smallest number of cuts in which this can be accomplished? One way of cutting the cube is to make a series of 6 cuts, 2 in each direction, while keeping the cube in one block as shown in Figure 1.2. But is it possible to use fewer cuts if the pieces can be rearranged between cuts? An example is given in Figure 1.3 where the second cut now cuts through more wood than it would have if we had not rearranged the pieces after the first cut.

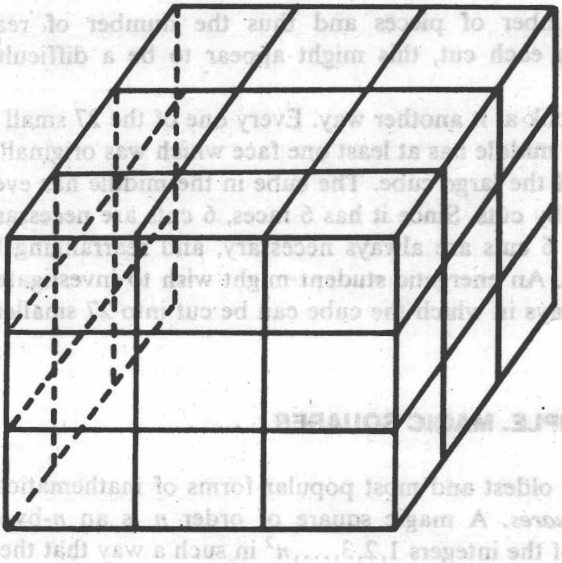

Figure 1.2

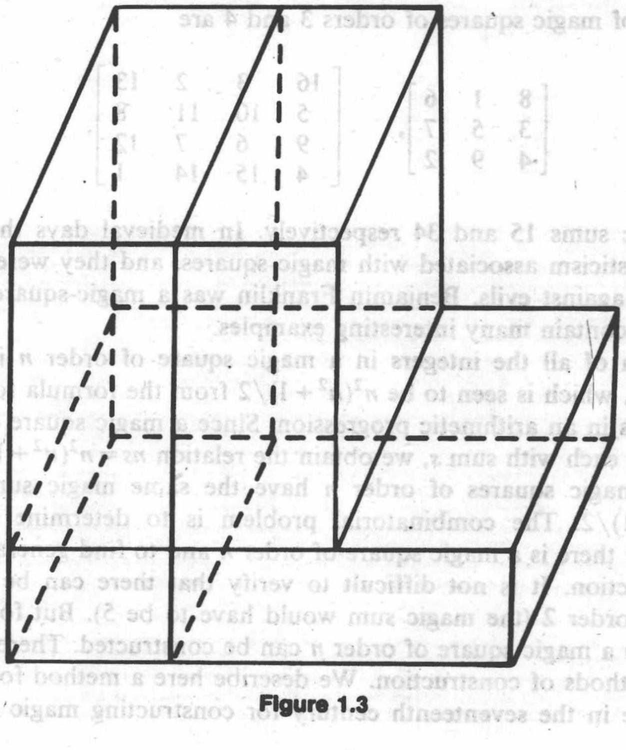

Figure 1.3

Since the number of pieces and thus the number of rearrangements increases with each cut, this might appear to be a difficult problem to analyze.

But let us look at it another way. Every one of the 27 small cubes except the one in the middle has at least one face which was originally part of one of the faces of the large cube. The cube in the middle has every one of its faces formed by cuts. Since it has 6 faces, 6 cuts are necessary to form it. Thus at least 6 cuts are always necessary, and rearranging between cuts does not help. An energetic student might wish to investigate the number of different ways in which the cube can be cut into 27 smaller cubes using only 6 cuts.

1.3 EXAMPLE. MAGIC SQUARES

Among the oldest and most popular forms of mathematical recreations are *magic squares*. A magic square of order n is an n-by-n array constructed out of the integers $1,2,3,\ldots,n^2$ in such a way that the sum s of the integers in each row, in each column, and in each of the two diagonals is the same. The number s is called the *magic sum* of the magic square. Examples of magic squares of orders 3 and 4 are

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}, \quad \begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}, \tag{1.1}$$

with magic sums 15 and 34 respectively. In medieval days there was a certain mysticism associated with magic squares, and they were worn for protection against evils. Benjamin Franklin was a magic-square fan, and his papers contain many interesting examples.

The sum of all the integers in a magic square of order n is $1+2+3+\cdots+n^2$, which is seen to be $n^2(n^2+1)/2$ from the formula for the sum of numbers in an arithmetic progression. Since a magic square of order n has n rows each with sum s, we obtain the relation $ns=n^2(n^2+1)/2$. Thus any two magic squares of order n have the same magic sum, namely $s=n(n^2+1)/2$. The combinatorial problem is to determine for which values of n there is a magic square of order n and to find general methods of construction. It is not difficult to verify that there can be no magic square of order 2 (the magic sum would have to be 5). But for all other values of n a magic square of order n can be constructed. There are many special methods of construction. We describe here a method found by de la Loubère in the seventeenth century for constructing magic squares of order n when n is odd. First a 1 is placed in the middle square of the top row. The successive integers are than placed in their natural order along a diagonal line which slopes upwards and to the right, with the following modifications.

(i) When the top row is reached, the next integer is put in the bottom row as if it came immediately above the top row.
(ii) When the right-hand column is reached, the next integer is put in the left-hand column as if it immediately succeeded the right-hand column.
(iii) When a square is reached which has already been filled or when the top right-hand square is reached, the next integer is placed in the square immediately below the last square which was filled.

The magic square of order 3 in (1.1) was constructed using de la Loubère's method, as was this magic square of order 5:

$$\begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix}. \qquad (1.2)$$

Methods for constructing magic squares of even orders different from 2 and other methods for constructing magic squares of odd order can be found in the book *Mathematical Recreations and Essays* by W. W. Rouse Ball, revised by H. S. M. Coxeter (New York: Macmillan, 1962, pp. 193–221).

1.4 EXAMPLE. THE 4-COLOR PROBLEM

Consider a map on a plane or on the surface of a sphere in which the countries are connected regions (thus the state of Michigan would not be allowed as a country of such a map). In order to be able to differentiate countries quickly, it is required to color them so that two countries which have a common boundary receive different colors. What is the smallest number of colors necessary to guarantee that every map can be so colored? Until recently, this was one of the most famous unsolved problems in mathematics. Its appeal to the layperson is due to the fact that it can be simply stated and understood. Except for the well-known angle-trisection problem, it has probably intrigued more amateur mathematicians than any other problem. First posed by Francis Guthrie about 1850 when he was a graduate student, it has also stimulated a large body of mathematical research. Indeed, an entire book, *The Four-Color Problem* by O. Ore (New York: Academic, 1967) is devoted to this problem and related problems. Some maps require 4 colors. An example is the map in Figure 1.4. Since each pair of the 4 countries of this map have a common boundary, it is clear that 4 colors are necessary to color the map. It was proven by P. J. Heawood in 1890 (see Chapter 11) that 5 colors are always enough to color any map. It is not too difficult to show (see Chapter 11) that it is impossible to have a map in the plane which has 5 countries, every pair of which have a boundary in common. Such a map, if it had existed, would have required 5 colors. In 1976 two mathematicians, K. Appel and W. Haken, astounded the mathematical community by announcing[2] that they had proven that any map in the plane could be colored with 4 colors. Their proof required about 1200 hours of computer calculations, nearly 10 billion separate, logical decisions!

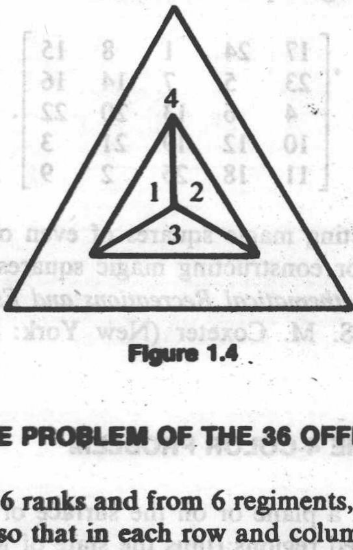

Figure 1.4

1.5 EXAMPLE. THE PROBLEM OF THE 36 OFFICERS

Given 36 officers of 6 ranks and from 6 regiments, can they be arranged in a 6-by-6 formation so that in each row and column there is one officer of each rank and one officer from each regiment? This problem, which was posed in the eighteenth century by the Swiss mathematician L. Euler (one of the most prolific mathematicians of all time) as a problem in recreational mathematics, has important repercussions in statistics, especially in the design of experiments (see Section 9.3). An officer can be designated by an ordered pair (i,j), where i denotes his rank $(i=1,2,\ldots,6)$ and j denotes his regiment $(j=1,2,\ldots,6)$. Thus the problem asks to *arrange the*

[2]*Every planar map is four colorable, Bulletin of The American Mathematical Society*, **82** (1976), 711–712.

36 ordered pairs (i,j) $(i=1,2,\ldots,6;\ j=1,2,\ldots,6)$ into a 6-by-6 array so that in each row and each column the integers $1,2,\ldots,6$ occur in some order in the first positions and in some order in the second positions of the ordered pairs. Such an array can be split into two 6-by-6 arrays, one corresponding to the first positions of the ordered pairs (the rank array) and the other to the second positions (the regiment array). Thus the problem can be stated: Do there exist two 6-by-6 arrays whose entries are taken from the integers $1,2,\ldots,6$ such that (i) in each row and in each column of these arrays the integers $1,2,\ldots,6$ occur in some order and (ii) when the two arrays are juxtaposed all of the 36 ordered pairs (i,j) $(i=1,2,\ldots,6;\ j=1,2,\ldots,6)$ occur? To make this concrete, suppose there are 9 officers of 3 ranks and from 3 different regiments. Then a solution for the corresponding problem is

$$\underset{\text{rank array}}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}} \cdot \underset{\text{regiment array}}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}} \rightarrow \underset{\text{juxtaposed array}}{\begin{bmatrix} (1,1) & (2,2) & (3,3) \\ (3,2) & (1,3) & (2,1) \\ (2,3) & (3,1) & (1,2) \end{bmatrix}}. \tag{1.3}$$

The rank and regiment arrays above are examples of what are called *Latin squares* of order 3. The following are Latin squares of orders 2 and 4:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}. \tag{1.4}$$

The two Latin squares of order 3 in (1.3) are called *orthogonal* because when there are juxtaposed all the 9 possible ordered pairs (i,j) with $i=1,2,3$ and $j=1,2,3$ result. We can thus rephrase Euler's question: Do there exist two orthogonal Latin squares of order 6? Euler investigated the more general problem of orthogonal Latin squares of order n. It is easy to verify that there is no pair of orthogonal Latin squares of order 2, since besides the Latin square of order 2 given in (1.4) the only other one is

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

and these are not orthogonal. Euler showed how to construct a pair of orthogonal Latin squares of order n whenever n is odd or has 4 as a factor. Notice that this does not include $n=6$. On the basis of many trials he concluded, but did not prove, that there was no pair of orthogonal Latin