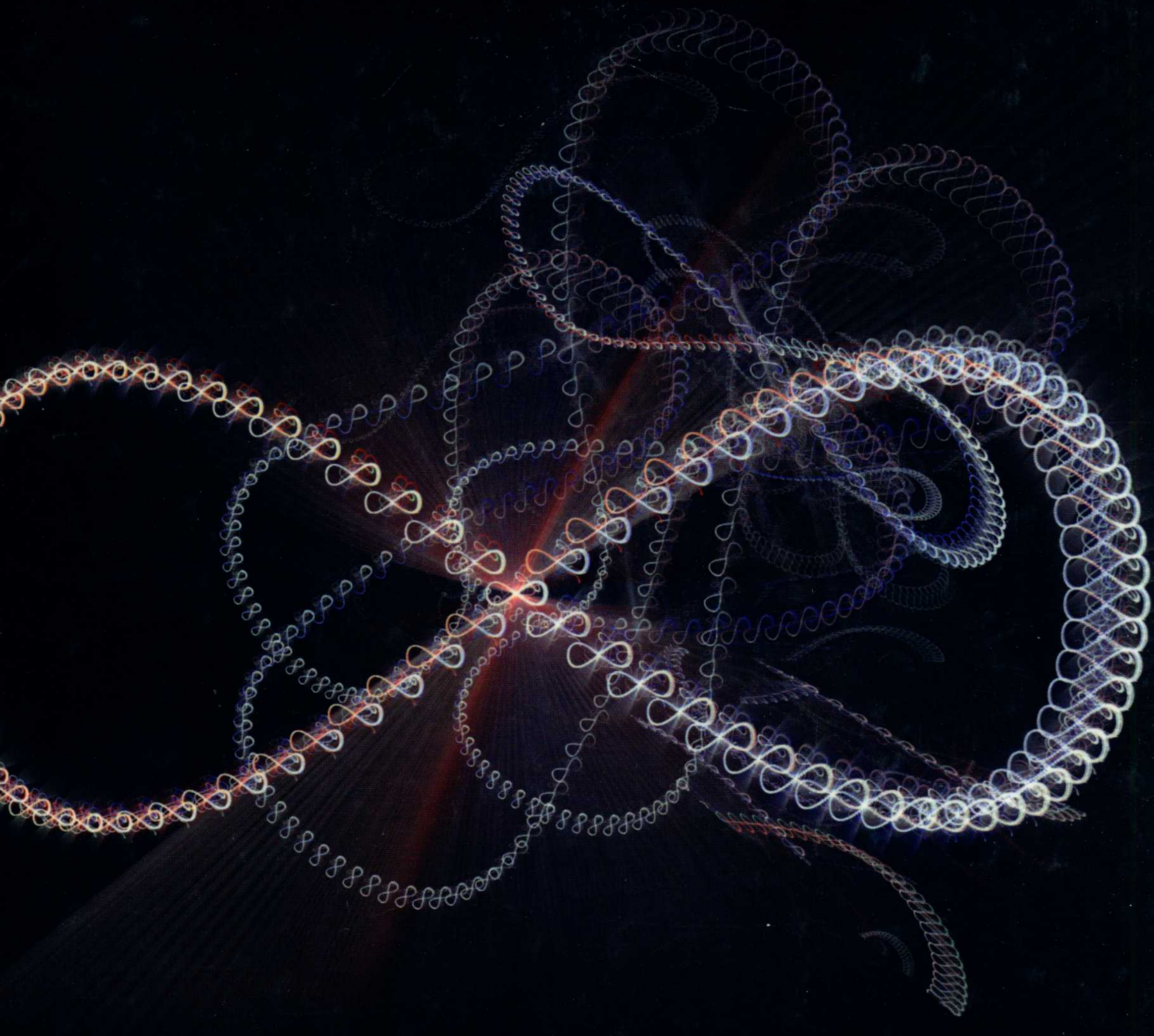
INFINITY

NEW RESEARCH FRONTIERS



EDITED BY Michael Heller

W. Hugh Woodin

MBRIDGE

Infinity

New Research Frontiers

Edited by

Michael Heller

Pontifical University of John Paul II

W. Hugh Woodin

University of California, Berkeley

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo, Mexico City

Cambridge University Press
32 Avenue of the Americas, New York, NY 10013-2473, USA

www.cambridge.org
Information on this title: www.cambridge.org/9781107003873

First published 2011

Printed in the United States of America

A catalog record for this publication is available from the British Library.

Library of Congress Cataloging in Publication data

Infinity : new research frontiers / edited by Michael Heller, W. Hugh Woodin.
p. cm.
Includes index.
ISBN 978-1-107-00387-3 (hardback)
1. Infinite – History. 2. Mathematics – Philosophy.
I. Heller, Michael. II. Woodin, W. H. (W. Hugh) III. Title.
QA9.I46 2011
111′.6–dc22 2010049374

ISBN 978-1-107-00387-3 Hardback

Infinity

New Research Frontiers

This interdisciplinary study of infinity explores the concept through the prism of mathematics and then offers more expansive investigations in areas beyond mathematical boundaries to reflect the broader, deeper implications of infinity for human intellectual thought. More than a dozen world-renowned researchers in the fields of mathematics, physics, cosmology, philosophy, and theology offer a rich intellectual exchange among various current viewpoints, rather than a static picture of accepted views on infinity.

The book starts with a historical examination of the transformation of infinity from a philosophical and theological study to one dominated by mathematics. It then offers technical discussions on the understanding of mathematical infinity. Following this, the book considers the perspectives of physics and cosmology: Can infinity be found in the real universe? Finally, the book returns to questions of philosophical and theological aspects of infinity.

Rev. Dr. Michael Heller is the Founder and Director of the Copernicus Center for Interdisciplinary Studies, Cracow. He is a Professor of Philosophy at the Pontifical University of John Paul II, Cracow, Poland; an Adjunct Member of the Vatican Observatory; and Ordinary Member of the Pontifical Academy of Sciences. Heller is the author of more than thirty books and nearly two hundred scientific papers.

Dr. W. Hugh Woodin is a Professor of Mathematics and the Chair of the Mathematics Department at the University of California, Berkeley. Professor Woodin has published numerous articles and books and is the managing editor of the *Journal of Mathematical Logic* and editor of *Mathematical Research Letters*, *Mathematical Logic Quarterly*, and *Electronic Research Announcements* (American Mathematical Society).

The infinite! No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite. . . .

David Hilbert (1862–1943)

Contributors

Wolfgang Achtner
Campus Minister and Assistant Professor (Privatdozent) of Systematic Theology, Justus Liebig University, Giessen; Assistant Professor (Privatdozent) of Systematic Theology, Johann Wolfgang Goethe University, Frankfurt; and Founder and Director of the Transscientia Institute for Interdisciplinary Research, Philosophy and Religion, Germany

Anthony Aguirre
Associate Professor of Physics, Department of Physics, University of California, Santa Cruz, United States

Marco Bersanelli
Professor of Astronomy and Astrophysics, Department of Physics, University of Milan, Italy

Enrico Bombieri
IBM von Neumann Professor of Mathematics, School of Mathematics, Institute for Advanced Study (New Jersey), United States

Harvey M. Friedman
Distinguished University Professor of Mathematics, Philosophy, and Computer Science, Department of Mathematics, The Ohio State University, Columbus, United States

David Bentley Hart
Visiting Professor and Robert J. Randall Chair in Christian Culture, Department of Theology, Providence College (Rhode Island), United States

Michael Heller
Founder and Director, Copernicus Center for Interdisciplinary Studies, Cracow; Professor of Philosophy, The Pontifical University of John Paul II, Cracow, Poland; Adjunct Member, Vatican Observatory, and Ordinary Member, Pontifical Academy of Sciences

Edward Nelson
Professor of Mathematics, Department of Mathematics, Princeton University (New Jersey), United States

Graham Oppy
Professor and Head, School of Philosophical, Historical and International Studies, Monash University at Clayton, Victoria, Australia

Carlo Rovelli
Senior Member, Academic Institute of France; Professor of Physics, University of the Mediterranean; Center for Theoretical Physics, National Center for Scientific Research (CNRS)-Luminy, Marseille, France

Rudy Rucker
Professor Emeritus, Mathematics and Computer Science, Departments of Mathematics and Computer Science, San José State University (California), United States

Robert John Russell
The Ian G. Barbour Professor of Theology and Science in Residence, The Graduate Theological Union; Founder and Director, The Center for Theology and the Natural Sciences, Berkeley (California), United States

Denys A. Turner
Horace Tracy Pitkin Professor of Historical Theology, Yale Divinity School, New Haven (Connecticut), United States

W. Hugh Woodin
Professor of Mathematics, Department of Mathematics, University of California, Berkeley, United States

Preface

Infinity: *New Research Frontiers* was developed to explore new research domains involving concepts of infinity in a technically rigorous, as well as interdisciplinary, context. It is the culmination of a creative research initiative that began with the international conference "New Frontiers in Research on Infinity," held August 18 to 20, 2006, in San Marino. The conference was co-organized by the Center for Theology and the Natural Sciences (CTNS), Berkeley, California,[1] and the John Templeton Foundation (JTF), Philadelphia, Pennsylvania,[2] and it was funded by a generous grant to CTNS from JTF with assistance from Euresis, Milan,[3] and the Republic of San Marino.[4] The invitation-only conference, whose theme was the concept and meaning of infinity in the multifaceted contexts of mathematics, physics, cosmology, philosophy, and theology, laid the intellectual groundwork for this volume.

Esteemed researchers in these diverse fields were invited to contribute to this book with the purpose of pursuing one of the "biggest questions" facing humankind: the notion of infinity. As the great German mathematician David Hilbert stated, as quoted in the epigraph at the beginning of this volume, "No other question has ever moved so profoundly the spirit of man." Infinity has usually been regarded as a "limiting concept," that is, a concept to which we arrive by extrapolating from what we know and what is limited and finite. However, some thinkers, following the example of Descartes, claim that infinity is a "primordial concept" and that all other concepts are derived from it. Thus, in this book, we ask the basic question: Is our world of everyday experience embedded in something transcending it?

Within, we offer our intellectual adventures with the concept of infinity, showing its multiform, interdisciplinary nature. The perspective adopted in the book is to show the tension between different viewpoints and opinions, presenting an academic "dispute in action" rather than a static panorama of accepted views. Previously residing in the realm of philosophical and theological speculations, considerations about infinity are

1 "Promoting the creative mutual interaction between contemporary *theology and the natural sciences*": http://www.ctns.org/.

2 "Supporting science, investing in the big questions": http://www.templeton.org/.

3 The Association for the Promotion of the Development of Culture and Scientific Work: http://www.euresis.org/.

4 See http://www.ctns.org/infinity/ for information about the symposium on which this book is based.

currently dominated by mathematics, and it is mathematics that influences our thinking about infinity in other domains.

Thus, first viewed through the prism of mathematics, infinity is then investigated more expansively in areas beyond mathematical boundaries to reflect the broader, deeper implications of the concept for human intellectual thought and explorations of our lives in the vastness of the universe: infinity in mathematics, in physics and cosmology, and in philosophy and theology.

Following the comprehensive introduction, which takes us through some of the aspects of infinity from the vantage points of various human pursuits, Part I presents a chapter showing the historical process of the expansion of investigations of infinity into other areas of human endeavor – infinity as a transformative concept in science and theology. Mathematics is powerfully represented in the book in a series of five mathematical chapters (divided into two parts) by prominent mathematicians, who present modern views of infinity. Part II contains two somewhat less technical discussions, and Part III contains three advanced studies contributing to our understanding of mathematical infinity. These include how infinity fits into current research in the foundations of mathematics and range from a prediction of imminent collapse to a prediction of complete vindication in the conception of the infinite.

Can infinity be found in the real universe? The question is far from trivial, and it is considered by prominent scientists in physics and cosmology in the four chapters presented in Part IV. Given the profundity of the subject matter, the editors devoted the remaining four chapters to exploring the inevitable historical, philosophical, and theological aspects of the vast notion of infinity in Part V. These reflections are based on Western metaphysics and the Judeo-Christian tradition and have no agenda other than to explore developments in these areas of human concern.

The integration of the somewhat disparate disciplines represented in this volume has resulted in an exploration of the concept and meaning of infinity that would not otherwise have been possible. Given the recent surge in activity among researchers trying to expand the mathematical and conceptual foundations of our understanding of the universe, we believe this book will appeal to a broad, educated readership. In addition, a volume of this sort makes it possible to present in one place a rather high level of technical information while rounding out the discussion of a profound conceptual and foundational question in a multidisciplinary format that would be inappropriate in a technical mathematics research volume. Thus, we hope this book will be of benefit to those without advanced mathematics and scientific backgrounds who are deeply interested in a topic that is typically explored only by specialists and who may find new inspiration in contemplating a boundless universe within these pages, which provide fascinating reading for all those who are not afraid of intellectual challenge.

***

Michael Heller, Founder and Director, Copernicus Center for Interdisciplinary Studies, Cracow; Professor of Philosophy, The Pontifical University of John Paul II, Cracow, Poland; Adjunct Member, Vatican Observatory, and Ordinary Member, Pontifical Academy of Sciences

and

W. Hugh Woodin, Professor of Mathematics, Department of Mathematics, University of California, Berkeley, United States

Acknowledgments

The editors wish to thank the co-organizers of the 2006 conference, “New Frontiers in Research on Infinity,” on which this volume is based: the Center for Theology and the Natural Sciences (CTNS), Berkeley, California, and the John Templeton Foundation (JTF), Philadelphia, Pennsylvania. In addition, we wish to thank JTF, Euresis, and the Republic of San Marino for co-sponsoring the conference.

While much of the conceptual structure of, and resources for, the conference and this book came initially from JTF senior staff and consultants, Robert John Russell, Director of CTNS, and his staff worked closely with JTF to develop and host the conference. Dr. Russell is also a contributor to this volume and, in addition, he worked with the co-editors during the writing and review process.

We also wish to thank Charles L. Harper, Jr., currently Chancellor for International Distance Learning and Senior Vice President, Global Programs, of the American University System, as well as President of Vision-Five.com Consulting, who served as one of the original project developers and the conference convener in his former role as Senior Vice President and Chief Strategist of JTF, working with Robert Russell of CTNS.

Hyung S. Choi, currently Director of Mathematical and Physical Sciences at JTF, assumed an integral role in developing the academic program for the symposium in conjunction with Charles Harper and Robert Russell.

Pamela M. Contractor, President and Director of Ellipsis Enterprises, Inc., working in conjunction with JTF and the volume editors, served as developmental editor of this book, along with Matthew P. Bond, Assistant Editor and Manager of Client Services at Ellipsis.

Finally, the editors acknowledge Lauren Cowles, Senior Editor, Mathematics and Computer Science at Cambridge University Press–New York, for supporting and overseeing this book project.

Contents

IV Perspectives on Infinity from Physics and Cosmology

V Perspectives on Infinity from Philosophy and Theology

Introduction

Rudy Rucker

A stimulating factor in discussions of infinity is that the concept arises in many different contexts: mathematics, physics, metaphysics, theology, psychology, and even the arts. The founder of modern set theory, Georg Cantor, was well aware of these distinctions, and he collapses them into three domains.

> The actual infinite arises in three contexts: *first* when it is realized in the most complete form, in a fully independent other-worldly being, *in Deo*, where I call it the Absolute Infinite or simply Absolute; *second* when it occurs in the contingent, created world; *third* when the mind grasps it *in abstracto* as a mathematical magnitude, number, or order type. I wish to make a sharp contrast between the Absolute and what I call the Transfinite, that is, the actual infinities of the last two sorts, which are clearly limited, subject to further increase, and thus related to the finite.[1]

Mathematical infinities occur as, for instance, the number of points on a continuous line, the size of the endless natural number sequence $1, 2, 3, \ldots$, or the class of all sets.

In physics, we encounter infinities when we wonder if there might be infinitely many stars, if the universe might last forever, or if matter might be infinitely divisible.

In metaphysical discussions of the Absolute, we can ask whether an ultimate entity must be infinite, whether lesser things can be infinite as well, and how the infinite relates to our seemingly finite lives.

The metaphysical questions carry over to the theological realm, and with an added emotional intensity. Theologians might, for instance, speculate about how a finite, created mind experiences an infinite God's love.

In the psychological domain, some might argue that it's impossible to talk coherently about infinity at all, whereas others report meditative mental perceptions of the infinite.

[1] Georg Cantor. 1980. *Gesammelte Abhandlungen,* p. 378. Berlin: Springer.. This translation is taken from my book, *Infinity and the Mind*, p. 9. Princeton: Princeton University Press, 2004. Robert John Russell also mentions this quote in his chapter in the present volume, "God and Infinity: Theological Insights from Cantor's Mathematics."

And, finally, in the arts, we seek to find representations of our looming intimations of the infinite, perhaps in paintings, or in music, poetry, or prose.

In the following remarks, I'll say a bit about the different kinds of infinity, with special attention to the contents of the essays gathered in this volume. I'll divide my remarks into four sections: (1) mathematical infinities, (2) physical infinities, (3) metaphysical and theological infinities, and (4) psychological and artistic infinities.

Mathematical Infinities

Enrico Bombieri's genial, discursive chapter, "The Mathematical Infinity," gives us a historical survey of many areas in which infinity has cropped up in mathematics, running from the Pythagoreans to the P and NP problem in computer science. Wolfgang Achtner's chapter, "Infinity as a Transformative Concept in Science and Theology," describes how the evolution of mathematical and physical notions of infinity has advanced in concert with our theological notions of infinity. I'll say more about Achtner's chapter in the section on metaphysical and theological infinities.

For now, I'll describe a high point of the history of mathematical infinity in my own words. Set theory, or the mathematical theory of infinity, was in large part created by Georg Cantor in the late 1800s. Cantor distinguishes between a specific set and the abstract notion of its size. In Cantor's theory, there's no contradiction or incoherence in having, say, two times a transfinite cardinal be the same transfinite cardinal. And, unlike finite sets, an infinite set can have the same cardinality as a proper subset of itself. Cantor calls these infinite number sizes "transfinite cardinals."

Cantor's celebrated theorem of 1873 shows that there are transfinite cardinals of strictly different sizes. Using a so-called diagonal argument, Cantor proved that the size of the set of whole numbers is strictly less than the size of the set of all points on a line. More generally, he showed that the cardinality of any set must be less than the cardinality of its power set, that is, the set that contains all the given set's possible subsets. Along with a principle known as the axiom of choice, the proof method of Cantor's theorem can be used to ensure an endless sequence of ever-larger transfinite cardinals.

The transfinite cardinals include aleph-null (the size of the set of whole numbers), aleph-one (the next larger infinity), and the continuum (the size of the set of points on a line). These three numbers are also written as $\aleph_0$, $\aleph_1$, and c. By definition, $\aleph_0$ is less than $\aleph_1$, and by Cantor's theorem, $\aleph_1$ is less than or equal to c. And we can continue on past $\aleph_1$ to such numbers as $\aleph_2$ and $\aleph_{\aleph_0}$.

The continuum problem is the question of which of the alephs is equal to the continuum cardinality c. Cantor conjectured that $c = \aleph_1$; this is known as Cantor's continuum hypothesis, or CH for short. The continuum hypothesis can also be thought of as stating that any set of points on the line must either be countable (of size less than or equal to $\aleph_0$) or have a size as large as the entire space (be of size c).

In the early 1900s a formalized version of Cantor's theory of infinite sets arose. This theory is known as ZF, which stands for "Zermelo-Fraenkel set theory." Informally speaking, the theory is often taken to include the axiom of choice.

The continuum hypothesis is known to be undecidable on the basis of the axioms in ZF. In 1940, the logician Kurt Gödel was able to show that ZF can't disprove CH, and in 1963, the mathematician Paul Cohen showed that ZF can't prove CH. Set theorists continue to explore ways to extend the ZF axioms in a reasonable way so as to resolve CH. In the early 1970s, Kurt Gödel suggested that CH may be false and that the true size of c could be the larger infinity $\aleph_2$. In 2001, the mathematician W. Hugh Woodin seemingly espoused this view as well.[2]

In one of his two chapters for the present volume, "The Realm of the Infinite," W. Hugh Woodin mounts farther than ever into the pinnacles of the infinite. Woodin and like-minded set theorists feel that Cantor's continuum problem may in fact be solvable, and that the answer is likely to have some relation to large cardinal axioms, which posit higher and higher levels of infinity.

Near the start of his chapter, Woodin makes the point that asserting the consistency of set theory is equivalent to asserting that certain destructive types of proofs will never be found to exist in the physical world. Thus, in this sense there is a direct, albeit subtle, connection between set theory and the physical world. Woodin feels that this connection lends some validity to the belief that the universe of set theory is real.

Some skeptics maintain that Cantor's continuum problem is, in fact, a meaningless question, akin to asking about, say, the fictional Frodo Baggins's precise height. In "The Realm of the Infinite," Woodin deploys a number of refined arguments for the reality and the objectivity of the continuum problem.

In particular, he wants to undermine what he calls the "generic-multiverse position," which suggests that, although we have many diverse models of set theory, there is really no one true model wherein something like the continuum hypothesis is definitely true or false. In the multiversal kind of view, CH is true in some models, false in others, and that's the end of it.

The perennial hope among set theorists is that if we can attain still higher conceptions of infinity, these insights may end up by shedding light on even such relatively low-level questions as the continuum problem. These novel kinds of infinity are collectively known as large cardinals. It may be that by extending set theory with some new axioms about large cardinals, we could narrow in on a more complete and satisfying theory.

Let me remark in passing that there is some similarity between a contemplative monk's quest for God and a set theorist's years-long and highly focused study of large cardinals.

Woodin formulates his analysis in terms of such highly advanced modern notions as his Ω Conjecture, the Inner Model Program, and his Set Theorist's Cosmological Principle. This is twenty-first-century mathematics, and a delight to behold, even if the details will lie beyond many of us.

Woodin is arguing for a maximally rich universe of set theory, in which new levels of surprise and creativity can be found at arbitrarily large levels. In his words, "It is a fairly common (informal) claim that the quest for truth about the universe of sets is analogous to the quest for truth about the physical universe. However, I am claiming an important distinction. While physicists would rejoice in the discovery that the conception of the physical universe reduces to the conception of some simple fragment or model, the set theorist rejects this possibility. By the very nature of its conception, the set of all truths

[2] W. Hugh Woodin. 2001. The continuum hypothesis, part I and part II. *Notices of the American Mathematical Society* (June/July and August).

of the transfinite universe (the universe of sets) cannot be reduced to the set of truths of some explicit fragment of the universe of sets."

Infinity isn't the only concept that lies on the interface between mathematics and philosophy. In his other startling contribution to this volume, "A Potential Subtlety Concerning the Distinction between Determinism and Nondeterminism," Woodin uses his brand of intellectual legerdemain to argue that there really is no coherent distinction between free will and determinism. The proofs are rigorous and subtle, drawing in deep results from recursion theory and nonstandard model theory.

Woodin's line of thought is similar to the following argument, which is drawn from computer science. Philosophers often suppose that there are only two options. *Either* we are deterministic and all of our decisions are predictable far in advance *or* our behavior is utterly capricious and essentially random. But there is a third way. A human being's behavior may indeed be generated by something like a mathematical algorithm, but it may be that the workings of the algorithm do not admit for any kind of shortcuts or speedups. That is, human behavior can be deterministic without being predictable.

Those not well versed in mathematical set theory sometimes imagine there to be a strong likelihood that our formal science of the infinite may contain an inconsistency. If, for instance, ZF set theory were to be inconsistent, then at some point we'd learn that the theory breaks down and begins "proving" things like 0=1. In this case, the theory would be useless.

Most professional set theorists develop a kind of sixth sense, whereby they feel themselves to be proving things about a Platonic world of actually infinite objects. One of my thesis advisors, Gaisi Takeuti, used to say, "Why would you believe in electrons or in a tiny village in Russia that you've never seen – yet deny the reality of $\aleph_1$ or of the set of all real numbers?" To a mathematician who's "looking" at the class of all sets every day, it seems quite evident that set theory is consistent – in much the same way that a physicist is sure that the laws of physics are consistent. A theory has a concrete model if and only if it is consistent.

In his chapter, "Concept Calculus: Much Better Than," Harvey Friedman, who has often worked in the field of proof theory, takes a novel approach to questions about the consistency or inconsistency of the standard theory or ZF set theory. How do we win the confidence of someone who hesitates to believe in a world of actually infinite sets? Friedman's new idea is that we might possibly model set theory in terms of the ordinary informal concepts of "better than" and "much better than." The chapter is an interesting tour de force, quite technical and demanding in its details. Friedman's intended program is to find further deep connections between logic and common sense, and this is to be commended.

Cantor was well aware that some people are in some sense blind to the possibility of infinity. As Cantor puts it: "The fear of infinity is a form of short-sightedness that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds."[3]

[3] I quote and translate this remark in my book, *Infinity and the Mind*, p. 43. The original quote appears in Georg Cantor, *Gesammelte Abhandlungen*, p. 374.

In his chapter, "Warning Signs of a Possible Collapse of Contemporary Mathematics," Edward Nelson speaks of "the strong emotions of loathing and oppression that the contemplation of an actual infinity arouses in me. It is the antithesis of life, of newness, of becoming – it is finished." Nelson proceeds to argue that the notion of infinity is somehow inconsistent. He takes an extreme finitist tack and presents a case that even the existence of very large finite sets is questionable. But, in the end, there's a certain circularity to any a priori arguments for or against the possibility of infinity.

Physical Infinities

The science of physical infinities is much less developed than the science of mathematical infinities. The main reason is simply that the status of physical infinities is quite undecided. In physics, one might look for infinities in space, time, divisibility, or dimensionality, and I'll discuss this in this section.

It is worth noting, however, that we are still conspicuously lacking in any physical application for the transfinite numbers of set theory. Along these lines, Georg Cantor hoped he could find an application for transfinite set theory in the realm of physics – at one time he proposed that ordinary matter might be made of aleph-null particles and that aether (which we might now term electromagnetic fields) might be made of aleph-one particles. Cantor conjectured that the matter and fields might be decomposable into meaningful pieces based on his notions of the accumulation points of infinite series. It would be a great day for set theorists if anything along the lines of such theories ever reached fruition.

Some of the Greeks speculated that space had to be infinite because the notion of an edge to space is incoherent. But, as Carlo Rovelli mentions in "Some Considerations on Infinity in Physics," if we view the space of our three-dimensional surface as curved into the hypersurface of a hypersphere, we're able to have a space that is both finite and unbounded.

Is our universe really shaped like that? In "Infinity and the Nostalgia of the Stars," Marco Bersanelli discusses how recent measurements of the cosmic microwave background indicate that the overall curvature of our space is very close to being that of a flat, Euclidean space, although the possibility still remains that our space might after all be a large hypersphere, or even a negatively curved space. Bersanelli couches the result in an amusing way:

> It is as if Eratosthenes in his famous measurement of the radius of the Earth in 250 BC was not able to measure any curvature: then his conclusion would have been that the Earth might be flat and infinite, or that its radius is greater than a given size compatible with the accuracy of his observation.
>
> (Bernaselli, Chapter 9)

Bersanelli makes another point that is not so well known. Even if we knew our space were to be precisely flat, we could not inevitably conclude that it was as infinite as an endless plane, for we might allow for the possibility that space might be multiply connected – like the surface of a torus. It is possible to find finitely large and multiply connected spaces that are, in fact, flat or negatively curved.