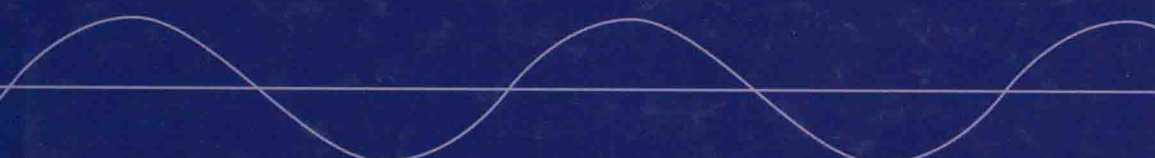


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Categorical Homotopy Theory

Emily Riehl

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Harvard University



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Categorical Homotopy Theory

This book develops abstract homotopy theory from the categorical perspective, with a particular focus on examples. Part I discusses two competing perspectives by which one typically first encounters homotopy (co)limits: either as derived functors definable when the appropriate diagram categories admit compatible model structures or through particular formulae that give the right notion in certain examples. Riehl unifies these seemingly rival perspectives and demonstrates that model structures on diagram categories are unnecessary. Homotopy (co)limits are explained to be a special case of weighted (co)limits, a foundational topic in enriched category theory. In Part II, Riehl further examines this topic, separating categorical arguments from homotopical ones. Part III treats the most ubiquitous axiomatic framework for homotopy theory – Quillen’s model categories. Here Riehl simplifies familiar model categorical lemmas and definitions by focusing on weak factorization systems. Part IV introduces quasi-categories and homotopy coherence.

EMILY RIEHL is a Benjamin Peirce Fellow in the Department of Mathematics at Harvard University and a National Science Foundation Mathematical Sciences Postdoctoral Research Fellow.

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To my students, colleagues, friends who inspired this work.

What we are doing is finding ways for *people* to understand and think about mathematics.

– William P. Thurston, “On proof and progress in mathematics”

Preface

The viewpoint taken by William Thurston's essay – that mathematical progress is made by advancing human understanding of mathematics and not only through the proof of new theorems – succinctly describes the character and focus of the course that produced this book. Although certain results appearing in this volume may surprise working homotopy theorists, the mathematical content of this text is not substantially new. Instead, the central value of this account derives from the more qualitative insights provided by its perspective. The theorems and topics discussed here illustrate how categorical formalisms can be used to organize and clarify a wealth of homotopical ideas.

The central project of homotopy theory, broadly defined, is to study the objects of a category up to a specified notion of “weak equivalence.” These weak equivalences are morphisms that satisfy a certain closure property vis-à-vis composition and cancellation that is also satisfied by the isomorphisms in any category – but weak equivalences are not generally invertible. In experience, it is inconvenient to work directly in the **homotopy category**, constructed by formally inverting these maps. Instead, over the years, homotopy theorists have produced various axiomatizations that guarantee that certain “point-set level” constructions respect weak equivalences and have developed models in which weak constructions behave like strict ones. By design, this patchwork of mathematical structures can be used to solve a wide variety of problems, but they can be rather complicated for the novice to navigate. The goal of this book is to use category theory to illuminate abstract homotopy theory and, in particular, to distinguish the formal aspects of the theory, principally having to do with enrichments, from techniques specific to the homotopical context.

The ordering of topics demands a few words of explanation. Rather than force the reader to persevere on good faith through pages of prerequisites, we

wanted to tell one of the most compelling stories right away. Following [22], we introduce a framework for constructing derived functors between categories equipped with a reasonable notion of weak equivalence that captures all the essential features of, but is much more general than, their construction in model category theory.

Why bother with this generalization? First, it exhibits the truth to the slogan that the weak equivalences are all that matter in abstract homotopy theory, showing that particular notions of cofibrations/fibrations and cofibrant/fibrant objects are irrelevant to the construction of derived functors – any notion will do. Second, and perhaps most important, this method for producing derived functors extends to settings, such as categories of diagrams of a generic shape, where appropriate model structures do not necessarily exist. In the culmination of the first part of this book, we apply this theory to present a uniform general construction of homotopy limits and colimits that satisfies both a local universal property (representing homotopy coherent cones) and a global one (forming a derived functor).

A further advantage of this approach, which employs the familiar two-sided (co)bar construction, is that it generalizes seamlessly to the enriched context. Any discussion of homotopy colimits necessarily encounters enriched category theory; some sort of topology on the ambient hom-sets is needed to encode the local universal property. These notes devote a fair amount of isolated attention to enriched category theory because this preparation greatly simplifies a number of later proofs. In general, we find it clarifying to separate the categorical aspects of homotopy theory from the homotopical ones. For instance, certain comparisons between models of homotopy colimits actually assert an isomorphism between the representing objects, not just the homotopy types. It is equally interesting to know when this is not the case.

Classical definitions of homotopy colimits, as in [10], are as **weighted colimits**. An ordinary colimit is an object that represents cones under a fixed diagram, whereas a homotopy colimit is an object representing “homotopy coherent” cones. The functor that takes an object in the diagram to the appropriately shaped homotopy coherent cone above it is called the **weight**. We believe that weighted limits and colimits provide a useful conceptual simplification for many areas of mathematics, and thus we begin the second part of this book with a thorough introduction, starting with the **Set**-enriched case, which already contains a number of important ideas. As we expect this topic to be unfamiliar, our approach is quite leisurely.

Our facility with enriched category theory allows us to be quite explicit about the role enrichment plays in homotopy theory. For instance, it is well known that the homotopy category of a simplicial model category is enriched over the homotopy category of spaces. Following [79], we present a general

framework that detects when derived functors and more exotic structures, such as weighted homotopy colimits, admit compatible enrichments. Enrichment over the homotopy category of spaces provides a good indication that these definitions are “homotopically correct.” Our formalism also allows us to prove that in an appropriate general context, total derived functors of left adjoints, themselves enriched over the homotopy category of spaces, preserve homotopy colimits.

We conclude this part with an interesting observation due to Michael Shulman: in the setting for these derived enrichment results, the weak equivalences can be productively compared with another notion of “homotopy equivalence” arising directly from the enrichment. Here we are using “homotopy” very abstractly; for instance, we do not require an interval object. Nonetheless, in close analogy with classical homotopy theory, the localization at the weak equivalences factors through the localization at the homotopy equivalences. Furthermore, the former homotopy category is equivalent to a restriction of the latter to the “fibrant-cofibrant” objects, between which these two notions of weak equivalence coincide.

After telling this story, we turn in the third part, perhaps rather belatedly, to the model categories of Daniel Quillen. Our purpose here is not to give a full account – this theory is well documented elsewhere – but rather to emphasize the clarifying perspective provided by weak factorization systems, the constituent parts in a model structure that are in some sense orthogonal to the underlying homotopical structure visible to the axiomatization of [22]. Many arguments in simplicial homotopy theory and in the development of the theory of quasi-categories take place on the level of weak factorization systems and are better understood in this context.

The highlight of this section is the presentation of a new variant of Quillen’s small object argument due to Richard Garner [31] that, at essentially no cost, produces functorial factorizations in cofibrantly generated model categories with significantly better categorical properties. In particular, we show that a cofibrantly generated simplicial model category admits a fibrant replacement monad and a cofibrant replacement monad that are simplicially enriched. Related observations have been made elsewhere, but we do not suspect that this precise statement appears in the literature.

The proofs of these results introduce ideas with broader applicability. A main theme is that the functorial factorizations produced by Garner’s construction have a much closer relationship to the lifting properties that characterize the cofibrations and fibrations in a model structure. Indeed, observations related to this “algebraic” perspective on the cofibrations and fibrations can be used to produce functorial factorizations for non-cofibrantly generated model structures [3].

Our construction of enriched functorial factorizations is complemented by a discussion of enriched lifting properties. There are notions of enriched weak factorization systems and enriched cofibrant generation, and these behave similarly to the familiar unenriched case. In the model structure context, this leads to a notion of an enriched model category that is reminiscent of but neither implies nor is implied by the usual axioms. This theory, which we believe is not found in the literature (the *nLab* aside), illuminates the distinction between the (classical) Quillen-type and Hurewicz-type model structures on the category of chain complexes over a commutative ring: the latter is an enrichment of the former. Indeed, the *same* sets of generating cofibrations and trivial cofibrations produce both model structures! We find it particularly interesting to note that the Hurewicz-type model structure, which is not cofibrantly generated in the traditional sense, is cofibrantly generated when this notion is enriched in the category of modules over the commutative ring (see [2]).

The section on model categories concludes with a brief exposition of Reedy category theory, which makes use of weighted limits and colimits to simplify foundational definitions. This chapter contains some immediate applications, proving that familiar procedures for computing homotopy limits and colimits in certain special cases have the same homotopy type as the general formulae introduced in Part I. Further applications of Reedy category theory follow later in our explorations of various “geometric” underpinnings of quasi-category theory.

In the final part of this book, we give an elementary introduction to quasi-categories, seeking, wherever possible, to avoid repeating things that are clearly explained in [49]. After some preliminaries, we use a discussion of homotopy coherent diagrams to motivate a translation between quasi-categories and simplicial categories, which are by now more familiar. Returning our attention to simplicial sets, we study isomorphisms within and equivalences between quasi-categories, with a particular focus on inverting edges in diagrams. The last chapter describes geometrical and 2-categorical motivations for definitions encoding the category theory of quasi-categories, presenting a number of not-yet-published insights of Dominic Verity. This perspective will be developed much more fully in [74, 75]. A reader interested principally in quasi-categories would do well to read Chapters 7, 11, and 14 first. Without this preparation, many of our proofs become considerably more difficult.

Finally, the very first topic is the author’s personal favorite: Kan extensions. Part of this choice has to do with Harvard’s unique course structure. The first week of each term is “shopping period,” during which students pop in and out of a number of courses prior to making their official selections. Anticipating a number of students who might not return, it seemed sensible to discuss a topic that is reasonably self-contained and of the broadest interest – indeed, significant applications appear throughout this text.

Prerequisites

An ideal student might have passing acquaintance with some of the literature on this subject: simplicial homotopy theory via [32, 55]; homotopy (co)limits via [10]; model categories via one of [24, 36, 38, 58, 65]; quasi-categories via [40, 49]. Rather than present material that one could easily read elsewhere, we chart a less-familiar course that should complement the insights of the experienced and provide context for the naïve student who might later read the classical accounts of this theory. The one prerequisite on which we insist is an acquaintance with and affinity for the basic concepts of category theory: functors and natural transformations; representability and the Yoneda lemma; limits and colimits; adjunctions; and (co)monads. Indeed, we hope that a careful reader with sufficient categorical background will emerge from this book confident that he or she fully understands each of the topics discussed here.

While the categorical prerequisites are essential, acquaintance with specific topics in homotopy theory is merely desired and not strictly necessary. Starting from Chapter 2, we occasionally use the language of model category theory to suggest the right context and intuition to those readers who have some familiarity with it, but these remarks are inessential. For particular examples appearing in the following, some acquaintance with simplicial sets in homotopy theory would also be helpful. Because these combinatorial details are essential for quasi-category theory, we give a brief overview in Chapter 15, which could be positioned earlier, were it not for our preference to delay boring those for whom this is second nature.

Dual results are rarely mentioned explicitly, except in cases where there are some subtleties involved in converting to the dual statement. In Chapters 1 and 2, we make casual mention of 2-categories before their formal definition – categories enriched in **Cat** – is given in Chapter 3. Note all 2-categories that appear are strict. Interestingly for a monograph devoted to the study of a weakened notion of equivalence between objects, we have no need for the weaker variants of 2-category theory.

Notational Conventions

We use boldface for technical terms that are currently being or will soon be defined and quotation marks for nontechnical usages meant to suggest particular intuition. Italics are for emphasis.

We write $\emptyset$ and $*$ for initial and terminal objects in a category. In a symmetric monoidal category, we also use $*$ to denote the unit object, whether or not the unit is terminal. We use $\mathbb{1}, 2, \dots$ for ordinal categories; for example, $\mathbb{2}$ is the

category $\bullet \rightarrow \bullet$ of the “walking arrow.” Familiar categories of sets, pointed sets, abelian groups, k -vector spaces, categories, and so on are denoted by **Set**, **Set_{*}**, **Ab**, **Vect_k**, **Cat**, and so on; **Top** should be a convenient category of spaces, as treated in Section 6.1. Generally, the objects of the category so denoted are suggested by a boldface abbreviation, and the morphisms are left unmentioned, assuming the intention is the obvious one.

We generally label the composite of named morphisms through elision but may use a $\cdot$ when the result would be either ambiguous or excessively ugly. The hom-set between objects x and y in a category $\mathcal{C}$ is most commonly denoted by $\mathcal{C}(x, y)$, although $\text{hom}(x, y)$ is also used on occasion. An underline, for example $\underline{\mathcal{C}}(x, y)$ or $\underline{\text{hom}}(x, y)$, signals that extra structure is present; the form this structure takes depends on what sort of enrichment is being discussed. In the case where the enrichment is over the ambient category itself, we frequently use exponential notation y^x for the internal hom-object. For instance, $\mathcal{D}^{\mathcal{C}}$ denotes the category of functors $\mathcal{C} \rightarrow \mathcal{D}$.

Natural transformations are most commonly denoted with a double arrow $\Rightarrow$ rather than a single arrow. This usage continues in a special case: a natural transformation $f \Rightarrow g$ between diagrams of shape $\mathbb{2}$, that is, between morphisms f and g , is simply a commutative square with f and g as opposing faces. The symbol $\rightrightarrows$ is used to suggest a parallel pair of morphisms, with common domain and codomain. Given a pair of functors $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$, use of a reversed turnstile $F \dashv G$ indicates that F is left adjoint to G .

Displayed diagrams should be assumed to commute, unless explicitly stated otherwise. The use of dotted arrows signals an assertion or hypothesis that a particular map exists. Commutative squares decorated with a $\lceil$ or a $\rfloor$ are pushouts or pullbacks, respectively. We sometimes use $\sim$ to decorate weak equivalences. The symbol $\cong$ is reserved for isomorphisms, sometimes simply denoted with an equality. The symbol $\simeq$ signals that the abutting objects are equivalent in whatever sense is appropriate, for example, homotopy equivalent or equivalent as quasi-categories.

Certain simplicial sets are given the following names: Δ^n is the standard (represented) n -simplex; $\partial\Delta^n$ is its boundary, the subset generated by non-degenerate simplices in degree less than n ; Λ_k^n is the subset with the k th codimension-one face also omitted. We follow the conventions of [32] and write d^i and s^j for the elementary simplicial operators (maps in Δ between $[n]$ and $[n - 1]$). The contrasting variance of the corresponding maps in a simplicial set is indicated by the use of lower subscripts — d_i and s_j — though whenever practical, we prefer instead to describe these morphisms as right actions by the simplicial operators d^i and s^j . This convention is in harmony with the Yoneda lemma: the map d^i acts on an n -simplex x of X , represented by a morphism $x: \Delta^n \rightarrow X$, by precomposing with $d^i: \Delta^{n-1} \rightarrow \Delta^n$.

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I consulted many sources while preparing these notes and wish to apologize for others deserving of mention that were inadvertently left out. More specific references appear throughout this text.

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The title and familiar content from Chapter 1 were of course borrowed from [50]. The presentation of the material in Chapters 2, 5, and 9–10 was strongly influenced by a preprint of Michael Shulman [79] and subsequent conversations with its author. His paper gives a much more thorough account of this story than is presented here; I highly recommend the original. The title of Chapter 3 was chosen to acknowledge its debt to the expository paper [46]. The content of Chapter 4 was surely absorbed by osmosis from my advisor, Peter May, whose unacknowledged influence can also be felt elsewhere. Several examples, intuitions, and observations appearing throughout this text can be found in the notes [18]; in the books [32, 36, 58]; or on the *n*Lab, whose collaborative authors deserve accolades. The material on weighted limits and colimits is heavily influenced by current and past members of the Centre of Australian Category Theory. I was first introduced to the perspective on model categories taken in Chapter 11 by Martin Hyland. Richard Garner shared many of the observations attributed to him in Chapters 12 and 13 in private conversation, and I wish to thank him for enduring endless discussions on this topic. It is not possible to overstate the influence Dominic Verity has had on this presentation of the material on quasi-categories. Any interesting unattributed results on that topic should be assumed to be due to him.

Finally, and principally, I wish to thank those who attended and inspired the course. This work is dedicated to them. Comments, questions, and observations from Michael Andrews, Omar Antolín Camarena, David Ayala, Tobias Barthel, Kestutis Cesnavicius, Jeremy Hahn, Markus Hausmann, Gijs Heuts, Akhil Mathew, Luis Pereira, Chris Schommer-Pries, Kirsten Wickelgren, Eric Wofsey, and Inna Zakharevich led to direct improvements in this text. Tobias Barthel and Moritz Groth made several helpful comments and caught a number of typos. Philip Hirschhorn, Barry Mazur, and Sophia Roosth were consulted on matters of style. I am grateful for the moral support and stimulating mathematical environment provided by the Boston homotopy theory community, particularly Mike Hopkins and Jacob Lurie at Harvard and Clark Barwick, Mark Behrens, and Haynes Miller at MIT. I would also like to thank Harvard

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