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Arithmetic of Diagonal Hypersurfaces over Finite Fields

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Notation and conventions

p: a rational prime number

$k = \mathbb{F}_q$: the finite field of q elements of $\text{char}(k) = p > 0$

$k^\times = \langle z \rangle$: the multiplicative group of k with a fixed generator z

$\bar{k}$: the algebraic closure of k

$(k^\times)^m := \{c^m \mid c \in k^\times\}$

$\Gamma = \text{Gal}(\bar{k}/k)$: the Galois group of $\bar{k}$ over k

$W = W(k)$: the ring of infinite Witt vectors over k

$K = K(k)$: the field of quotients of W

ν: a p-adic valuation of $\overline{\mathbb{Q}}_p$ normalized by $\nu(q) = 1$

F: the Frobenius morphism

V: the Verschiebung morphism

Φ: the Frobenius endomorphism

m and n: positive integers such that $m \geq 3, (m,p) = 1$ and $n \geq 1$

ℓ: a prime such that $(\ell, m) = 1$

$\mathbb{Q}_\ell$: the field of ℓ-adic rationals

$\mathbb{Z}_\ell$: the ring of ℓ-adic integers

$|\ \ |_\ell^{-1}$: the ℓ-adic valuation of $\mathbb{Q}$ normalized by $|\ell|_\ell^{-1} = \ell$

$|x|$: the absolute value of $x \in \mathbb{R}$

$L = \mathbb{Q}(\zeta)$: the m-th cyclotomic field over $\mathbb{Q}$ where $\zeta = e^{2\pi i/m}$

$G = \text{Gal}(L/\mathbb{Q})$: the Galois group of L over $\mathbb{Q}$, which is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^\times$

$\phi(m)$: the Euler function

$\mathbf{c} = (c_0, c_1, \ldots, c_{n+1}) \in \underbrace{k^\times \times \cdots \times k^\times}_{n+2 \text{ copies}}$: the twisting vector

$\mathcal{V} = \mathcal{V}_n^m(\mathbf{c})$: the diagonal hypersurface $\sum_{i=0}^{n+1} c_i X_i^m = 0 \subset \mathbb{P}_k^{n+1}$ with the twisting vector $\mathbf{c}$ of degree m and dimension n

$\mathcal{X} = \mathcal{V}_n^m(\mathbf{1})$: the Fermat variety $\sum_{i=0}^{n+1} X_i^m = 0 \subset \mathbb{P}_k^{n+1}$ of degree m and dimension n with the trivial twist $\mathbf{c} = \mathbf{1}$

μ_m: the group of m-th roots of unity in $\mathbb{C}$ (or in $\bar{k}$)

$\mathfrak{G} = \mathfrak{G}_n^m = \mu_m^{n+2}/\Delta$: a subgroup of the automorphism group $\mathrm{Aut}(\mathcal{V})$ of $\mathcal{V}$

$\hat{\mathfrak{G}}$: the character group of $\mathfrak{G}$

$\mathfrak{A} = \mathfrak{A}_n^m$: the set of all characters $\mathbf{a} = (a_0, a_1, \ldots, a_{n+1}) \in \hat{\mathfrak{G}}$ such that

$$a_i \in \mathbb{Z}/m\mathbb{Z}, \qquad a_i \not\equiv 0 \pmod{m}, \qquad \text{and} \qquad \sum_{i=0}^{n+1} a_i \equiv 0 \pmod{m}.$$

For $\mathbf{a} = (a_0, a_1, \ldots, a_{n+1}) \in \mathfrak{A}_n^m$,

$\|\mathbf{a}\| = \sum_{i=0}^{n+1} \langle \frac{a_i}{m} \rangle - 1$ where $\langle x \rangle$ is the fractional part of $x \in \mathbb{Q}$

$p_{\mathbf{a}}$: the projector defined in Definition 3.1

$j(\mathbf{a})$: a Jacobi sum of dimension n and degree m

$\mathfrak{J}(\mathbf{c}, \mathbf{a})$: a twisted Jacobi sum of dimension n and degree m

$\tilde{\mathbf{a}}$: an induced character in $\mathfrak{A}_{n+d}^m$ for some $d \geq 1$

$j(\tilde{\mathbf{a}})$: an induced Jacobi sum of an appropriate dimension and degree m

$\mathfrak{J}(\tilde{\mathbf{c}}, \tilde{\mathbf{a}})$: an induced twisted Jacobi sum of an appropriate dimension and degree m

$A = [\mathbf{a}]$: the $(\mathbb{Z}/m\mathbb{Z})^\times$-orbit of $\mathbf{a}$

$p_A = [\mathbf{a}] = \sum_{\mathbf{a} \in A} p_{\mathbf{a}}$

$\tilde{A} = [\tilde{\mathbf{a}}]$: the $(\mathbb{Z}/m\mathbb{Z})^\times$-orbit of $\tilde{\mathbf{a}}$

$\mathcal{M}_A$: a Fermat motive of degree m and dimension n

$\mathcal{V}_A$: a twisted Fermat motive of degree m and dimension n

$\mathcal{M}_{\tilde{A}}$: an induced Fermat motive of degree m and an appropriate dimension

$\mathcal{V}_{\tilde{A}}$: an induced twisted Fermat motive of degree m and an appropriate dimension

$\#S$: the cardinality (resp. order) of a set (resp. group) S

$\mathfrak{B}_n^m = \{\mathbf{a} \in \mathfrak{A}_n^m \mid \mathfrak{J}(\mathbf{c}, \mathbf{a}) = q^{n/2}\}$ with n even

$\overline{\mathfrak{B}}_n^m = \{\mathbf{a} \in \mathfrak{A}_n^m \mid \mathfrak{J}(\mathbf{c}, \mathbf{a})/q^{n/2} = \text{a root of unity in } L\}$ with n even

$\mathfrak{C}_n^m = \overline{\mathfrak{B}}_n^m \setminus \mathfrak{B}_n^m$

$\mathfrak{D}_n^m = \mathfrak{A}_n^m \setminus \mathfrak{B}_n^m$

$O(\mathfrak{C}_n^m)$: the set of $(\mathbb{Z}/m\mathbb{Z})^\times$-orbits in $\mathfrak{C}_n^m$

$O(\mathfrak{D}_n^m)$: the set of $(\mathbb{Z}/m\mathbb{Z})^\times$-orbits in $\mathfrak{D}_n^m$

$\varepsilon_d(\mathcal{V}_k) = \#O(\mathfrak{C}_n^m)$

$\lambda_d(\mathcal{V}_k) = \#O(\mathfrak{D}_n^m)$

$\delta_d(\mathcal{V}_k) = \varepsilon_d(\mathcal{V}_k) + \lambda_d(\mathcal{V}_k)$

Let M be a Γ-module where $\Gamma = \mathrm{Gal}(\bar{k}/k)$ with the Frobenius generator Φ.

M^Γ: the kernel of the map $\Phi - 1 : M \to M$

M_Γ: the cokernel of the map $\Phi - 1 : M \to M$

M_{tors}: the torsion subgroup of M

$\mathcal{O}$: the structure sheaf of $\mathcal{V}$ and $\mathcal{X}$

Ω: the sheaf of differentials on $\mathcal{V}$ and $\mathcal{X}$

$W\Omega$: the sheaf of de Rham–Witt complexes on $\mathcal{V}$ and $\mathcal{X}$

$\mathbb{G}_m$: the multiplicative group scheme

$\mathbb{G}_a$: the additive group scheme

Arithmetical invariants of $\mathcal{V}$ and $\mathcal{X}$ are rather sensitive to the fields of definition. Whenever the fields of definition are to be specified, subscripts are adjoined to the objects in question. For instance,

$\rho_r(\mathcal{V}_k)$ (resp. $\rho_r(\mathcal{V}_{\bar{k}})$: the r-th combinatorial Picard number of $\mathcal{V}$ defined over k (resp. $\bar{k}$)

$\rho_r'(\mathcal{V}_k)$ (resp. $\rho_r'(\mathcal{V}_{\bar{k}})$): the dimension of the subspace of $H^{2r}(\mathcal{V}_{\bar{k}}, \mathbb{Q}_\ell(r))$ generated by algebraic cycles of codimension r on $\mathcal{V}$ defined over k (resp. $\bar{k}$) where ℓ is a prime $\neq p$, which we call the r-th (geometric) Picard number of $\mathcal{V}$ defined over k (resp. $\bar{k}$)

$\mathrm{Br}^r(\mathcal{V}_k)$ (resp. $\mathrm{Br}^r(\mathcal{V}_{\bar{k}})$): the r-th “Brauer” group of $\mathcal{V}$ over k (resp. $\bar{k}$)

Contents

Introduction

Let $X = X_k$ be a smooth projective algebraic variety of dimension n defined over a finite field $k = \mathbb{F}_q$ of characteristic p. The zeta-function of X (relative to k) has the form

$$Z(X, q^{-s}) = \frac{P_1(X, q^{-s})P_3(X, q^{-s}) \ldots P_{2n-1}(X, q^{-s})}{P_0(X, q^{-s})P_2(X, q^{-s}) \ldots P_{2n}(X, q^{-s})}$$

where $P_i(X, T) \in 1 + T\mathbb{Z}[T]$ for every i, $0 \leq i \leq 2n$, and has reciprocal roots of absolute value $q^{i/2}$. Taking i equal to an even integer $2r$, we see that for any integer r between 0 and n

$$Z(X, q^{-s}) \sim \frac{C_X(r)}{(1 - q^{r-s})^{\rho_r(X)}} \quad \text{as } s \to r$$

where $C_X(r)$ is some rational number and $\rho_r(X)$ is an integer (called the r-th combinatorial Picard number of $X = X_k$). In this book, we obtain information about these two numbers for algebraic varieties that are especially simple.

There are standard conjectural descriptions of the numbers $\rho_r(X)$ and $C_X(r)$ that connect them with arithmetic and geometric invariants of X. Let $\bar{k}$ be an algebraic closure of k and let $X_{\bar{k}} := X \times_k \bar{k}$ be the base change of X from k to $\bar{k}$. Let ℓ be any prime different from $p = \mathrm{char}(k)$. Let $\rho'_{r,\ell}(X)$ denote the dimension of the subspace of the ℓ-adic étale cohomology group $H^{2r}(X_{\bar{k}}, \mathbb{Q}_\ell(r))$, generated by algebraic cycles of codimension r on X defined over k, and let

$$\rho'_r(X) := \max_{\ell \neq p} \rho'_{r,\ell}(X).$$

(The numbers $\rho'_{r,\ell}(X)$ are in fact presumed to be independent of the choice of the prime ℓ.) We call $\rho'_r(X)$ the r-th Picard number of $X = X_k$. It is known that $\rho'_r(X) \leq \rho_r(X)$, and one conjectures that they are in fact equal:

Conjecture 0.1 (The Tate Conjecture) *With the definitions above, we have*

$$\rho_r(X) = \rho'_r(X).$$

This is known to hold in a number of special cases (rational surfaces, Abelian surfaces, products of two curves, certain Fermat hypersurfaces, etc.)

Picard numbers are, of course, very sensitive to the field of definition. In various contexts we will want to compare the Picard number of a variety X over k to the Picard number of its base change to extensions of k. As one runs over bigger and bigger finite extensions of k, the combinatorial Picard number eventually stabilizes. We will refer to the latter number as the r-th (combinatorial) *stable* Picard number of X and denote it by $\overline{\rho}_r(X)$.

As for the rational number $C_X(r)$, a series of conjectures has been formulated by Lichtenbaum [Li84, Li87, Li90] and Milne [Mil86, Mil88] (see also Etesse [Et88]). (The conjectures concern the existence of "motivic cohomology" and in particular of certain complexes of étale sheaves $\mathbb{Z}(r)$.)

CONJECTURE 0.2 (THE LICHTENBAUM–MILNE CONJECTURE) *Assume that the complex $\mathbb{Z}(r)$ exists and that the Tate conjecture holds for $X = X_k$. Then*

$$C_X(r) = \pm\chi(X, \mathbb{Z}(r)) \cdot q^{\chi(X,\mathcal{O},r)}$$

where

$$\chi(X, \mathcal{O}, r) := r\chi(X, \mathcal{O}_X) - (r-1)\chi(X, \Omega_X^1) + \cdots \pm \chi(X, \Omega_X^{r-1})$$

and $\chi(X, \mathbb{Z}(r))$ is the Euler–Poincaré characteristic of the complex $\mathbb{Z}(r)$.

For surfaces, this formula is equivalent to the Artin–Tate formula, which is known to be true whenever the Tate conjecture holds. For higher dimensional varieties, the conjectural formula is known to hold only in some special cases. Therefore, providing examples related to this conjecture seems to be of considerable interest.

The purpose of these notes is to offer a testing ground for the Lichtenbaum–Milne conjecture for diagonal hypersurfaces, explicitly evaluating the special values of zeta-functions at integral arguments. This is done by passing to the twisted Fermat motives associated to such varieties. Our investigation is both theoretical and numerical; the results of our computations are recorded in Appendix A.

We now proceed to set up the case we want to investigate. Let m and n be integers such that $m \geq 3$, $(p, m) = 1$ and $n \geq 1$. Let $\mathbf{c} = (c_0, c_1, \cdots, c_{n+1})$ be a vector where $c_i \in k^\times$ for each $i = 0, 1, \ldots, n+1$, and let $\mathcal{V} = \mathcal{V}_n^m(\mathbf{c}) \subset \mathbb{P}_k^{n+1}$ denote the diagonal hypersurface of dimension n and of degree m defined over $k = \mathbb{F}_q$ given by the equation

$$c_0X_0^m + c_1X_1^m + \cdots + c_{n+1}X_{n+1}^m = 0. \qquad (*)$$

We denote by $\mathcal{X} := \mathcal{V}_n^m(\mathbf{1})$ the Fermat hypersurface of dimension n and of degree m defined by the equation $(*)$ with $\mathbf{c} = (1, 1 \cdots, 1) = \mathbf{1}$. We call the vector $\mathbf{c}$ a *twisting* vector. Note that the vector $\mathbf{c} = (c_0, c_1, \ldots, c_{n+1})$ is only defined up to multiplication by a non-zero constant, and further, that changing any of the coefficients by an element in $k^\times$ which is an m^{th} power gives an isomorphic variety. We will call two such choices for $\mathbf{c}$ equivalent. We will denote the set of all vectors $\mathbf{c} = (c_0, \ldots, c_{n+1})$, considered up to equivalence, by $\mathcal{C}$.

Throughout the book, we impose the hypothesis that k contains all the m-th roots of unity, which is equivalent to the condition that $q \equiv 1 \pmod{m}$.

The diagonal hypersurface $\mathcal{V} = \mathcal{V}_n^m(\mathbf{c})$ is a complete intersection, and its cohomology groups are rather simple (cf. Deligne [De73], Suwa [Su93]). Its geometry and arithmetic are closely connected to those of the Fermat hypersurface, $\mathcal{X} = \mathcal{V}_n^m(\mathbf{1})$. In fact, the eigenvalues of the Frobenius endomorphism for $\mathcal{X}$ are Jacobi sums, and those for $\mathcal{V}$ are *twisted* Jacobi sums, that is, Jacobi sums multiplied by some m-th root of unity. Furthermore, the geometric and topological invariants of $\mathcal{V}$, such as the Betti numbers, the (i, j)-th Hodge numbers, the slopes and the dimensions and heights of formal groups are independent of the twisting vectors $\mathbf{c}$ for the defining equation for $\mathcal{V}$, and therefore coincide with the corresponding quantities for $\mathcal{X}$. By contrast, arithmetical invariants of $\mathcal{V}$ (that are sensitive to the fields of definition), such as the Picard number, the group of algebraic cycles, and the intersection matrix, differ from the corresponding quantities for $\mathcal{X}$. Relations between these arithmetical invariants of $\mathcal{V}$ and the corresponding invariants of $\mathcal{X}$ are one of our main themes.

To understand the arithmetic of a diagonal hypersurface $\mathcal{V} = \mathcal{V}_n^m(\mathbf{c})$ of dimension n and degree m with twist $\mathbf{c}$, we use the natural group action to associate to it a family of motives which correspond to a particularly natural decomposition of the cohomology of $\mathcal{V}$, which we call the *motivic decomposition.* We call these (not necessarily indecomposable) motives *twisted Fermat motives*, and the direct sum of these motives is the motive attached to $\mathcal{V}$ itself. The arithmetic of these motives "glues together" to form the arithmetic of $\mathcal{V}$.

Let $\mathcal{V}_A$ denote a twisted Fermat motive. We say that $\mathcal{V}_A$ is *supersingular* if the Newton polygon of $\mathcal{V}_A$ has a pure slope $n/2$; $\mathcal{V}_A$ is *ordinary* if the Newton polygon of $\mathcal{V}_A$ coincides with the Hodge polygon of $\mathcal{V}_A$; and $\mathcal{V}_A$ is *of Hodge–Witt type* if the Hodge–Witt cohomology group $H^{n-i}(\mathcal{V}_A, W\Omega^i)$ is of finite type for every i, $0 \le i \le n$. (If $\mathcal{V}_A$ is ordinary, then it is of Hodge–Witt type, but the converse is not true.) Then passing to diagonal hypersurfaces $\mathcal{V}$, we say that $\mathcal{V}$ is *supersingular, ordinary,* and *of Hodge–Witt type* if every twisted Fermat motive $\mathcal{V}_A$ is supersingular, ordinary, and of Hodge–Witt type, respectively. Note that these properties are not disjoint at the motivic level (that is, motives can be ordinary and supersingular at the same time).

The set of all diagonal hypersurfaces has a rather elaborate *inductive structure*, relating hypersurfaces of fixed degree and varying dimension. We focus on two types of these: the first relating hypersurfaces of dimension n and $n+2$, and the second relating hypersurfaces of dimensions $n+1$ and $n+2$. This inductive structure is independent of the twisting vectors of the defining equation for $\mathcal{V}$. As before, the inductive structure can be considered at the motivic level, and the arithmetic and geometry of motives are closely related to those of their induced motives of higher dimension. Cohomological realizations of these structures shed light, for instance, on the Tate conjecture and on special values of (partial) zeta-functions. (For details, see Chapter 4 below.) This inductive structure also plays a major role in the work of Ran and Shioda on the Hodge conjecture for complex Fermat hypersurfaces (see [Ran81] and [Sh79a, Sh79b], for example).

For diagonal hypersurfaces $\mathcal{V} = \mathcal{V}_n^m(\mathbf{c})$ of odd dimension $n = 2d+1$, the Tate conjecture is trivially true (Milne [Mil86]). For diagonal hypersurfaces of dimension $n = 2$, the Tate conjecture can be proved for any twist $\mathbf{c}$ over k on the basis of the results of Tate [Ta65] and Shioda and Katsura [SK79] for Fermat surfaces $\mathcal{X}_2^m$ over k. We obtain the following result.

THEOREM 0.3
Let $\mathcal{V} = \mathcal{V}_n^m(\mathbf{c})$ be a diagonal hypersurface with twist $\mathbf{c}$ and let $\mathcal{X} = \mathcal{V}_n^m(\mathbf{1})$ be the Fermat variety, both of degree m and dimension $n = 2d$ over $k = \mathbb{F}_q$. Let $\rho_d(\mathcal{V})$ and $\rho_d(\mathcal{X})$ denote the d-th combinatorial Picard number of $\mathcal{V}$ and $\mathcal{X}$, respectively, and let $\overline{\rho}_d(\mathcal{V})$ and $\overline{\rho}_d(\mathcal{X})$ be the corresponding stable combinatorial Picard numbers. Then the following assertions hold:

1. *The combinatorial stable Picard numbers are given by*

$$\overline{\rho}_d(\mathcal{V}) = \overline{\rho}_d(\mathcal{X}) = 1 + \sum B_n(\mathcal{V}_A)$$

where the sum runs over all supersingular twisted Fermat motives $\mathcal{V}_A$, and $B_n(\mathcal{V}_A)$ denotes the n-th Betti number of $\mathcal{V}_A$.

2. *Assume that m is prime, $m > 3$. Then*

$$\rho_d(\mathcal{X}_k) = \overline{\rho}_d(\mathcal{V}).$$

That is, the actual d-th combinatorial Picard number of $\mathcal{X}_k$ is stable.

3. *Assume that m is prime, $m > 3$. Then*

$$\rho_d(\mathcal{V}_k) \leq \rho_d(\mathcal{X}_k).$$

Furthermore, the following are equivalent:

(a) *$\mathcal{V}_k$ and $\mathcal{X}_k$ are isomorphic*

(b) *$\rho_d(\mathcal{V}_k) = \rho_d(\mathcal{X}_k)$*

(c) **c** *is equivalent to the trivial twist* **1**.

Part 3 is false in general for composite m: for some values of m, one can find twists **c** such that $\rho_d(\mathcal{V}_k) > \rho_d(\mathcal{X}_k)$. One can also find non-trivial twists such that $\rho_d(\mathcal{V}_k) = \rho_d(\mathcal{X}_k)$. See section A.3.

Shioda [Sh82a] has obtained a closed formula for the stable Picard number for surfaces of prime degree: if $n = 2$, m is a prime, and $p \equiv 1 \pmod{m}$ then:

$$\overline{\rho}_1(\mathcal{V}) = 1 + 3(m-1)(m-2).$$

Similar formulas hold for higher-dimensional hypersurfaces.

PROPOSITION 0.4
Using definitions and notation as above,

1. *when* $n = 4$, m *is prime, and* $p \equiv 1 \pmod{m}$,
$$\overline{\rho}_2(\mathcal{V}) = 1 + 5(m-1)(3m^2 - 15m + 20),$$
2. *when* $n = 6$, m *is prime, and* $p \equiv 1 \pmod{m}$,
$$\overline{\rho}_3(\mathcal{V}) = 1 + 5 \cdot 7(m-1)(3m^3 - 27m^2 + 86m - 95).$$

When m is prime and $p \equiv 1 \pmod{m}$, Shioda's method allows such formulas to be computed for any specific even dimension. (See Appendix B for the details.) Similar methods allow one to get formulas that hold for more general degrees. Of course, these formulas only give the stable Picard numbers. When there is a non-trivial twist or when m is composite, determining the actual Picard number over $\mathbb{F}_q$ is much more delicate. We have computed many of these—see the tables in Appendix A.

Given a vector $\mathbf{c} = (c_0, c_1, \ldots, c_{n+1})$ and a character $\mathbf{a} = (a_0, a_1, \ldots a_{n+1})$, we define, in the usual way, $\mathbf{c}^{\mathbf{a}} := c_0^{a_0} c_1^{a_1} \cdots c_{n+1}^{a_{n+1}}$. We say **c** is *extreme* if we have $\mathbf{c}^{\mathbf{a}} \notin (k^\times)^m$ for any $\mathbf{a} = (a_0, a_1, \ldots, a_{n+1}) \in \mathfrak{A}_n^m$ with $j(\mathbf{a}) = q^d$. One reason extreme twists are interesting is the following observation.

THEOREM 0.5
Let $\mathcal{V} = \mathcal{V}_n^m(\mathbf{c})$ *be a diagonal hypersurface of dimension* $n = 2d$ *and prime degree* $m > 3$ *over* $k = \mathbb{F}_q$. *Suppose that* **c** *is extreme. Then the Tate conjecture holds for* $\mathcal{V}_k$, *and we have*

$$\rho_d'(\mathcal{V}_k) = \rho_d(\mathcal{V}_k) = 1.$$

In the case of an extreme twist, one can also determine the intersection pairing on the (one dimensional!) image of the d-th Chow group in the cohomology.

For general diagonal hypersurfaces $\mathcal{V} = \mathcal{V}_m^n(\mathbf{c})$, we can use the results of Ran [Ran81], Shioda [Sh79a, Sh79b, Sh83b] and Tate [Ta65] to establish the validity of the Tate conjecture in the following cases.

PROPOSITION 0.6
Let $\mathcal{V} = \mathcal{V}_n^m(\mathbf{c})$ be a diagonal hypersurface of dimension $n = 2d > 2$ and degree $m > 3$ with twist $\mathbf{c}$ over a finite field $\mathbb{F}_q$ with $q = p^j \equiv 1 \pmod m$. Then the Tate conjecture holds in the following cases:

1. *m is prime, any n, and $p \equiv 1 \pmod m$.*
2. *$m \leq 20$, any n, and $p \equiv 1 \pmod m$.*
3. *$m = 21$, $n \leq 10$, and $p \equiv 1 \pmod m$.*
4. *m and n arbitrary and there exists j such that $p^j \equiv -1 \pmod m$ (equivalently, $\mathcal{V}$ is supersingular).*

Since diagonal hypersurfaces $\mathcal{V} = \mathcal{V}_n^m(\mathbf{c})$ are complete intersections, their zeta-functions have the form:

$$Z(\mathcal{V}, T) = \frac{Q(\mathcal{V}, T)^{(-1)^{n+1}}}{\prod_{i=0}^{n}(1 - q^i T)}.$$

In our case, $Q(\mathcal{V}, T)$ is a polynomial of degree $\frac{m-1}{m}\{(m-1)^{n+1} + (-1)^{n+2}\}$ with integral coefficients, which factors over $\mathbb{C}$ as

$$Q(\mathcal{V}, T) = \prod_{\mathbf{a} \in \mathfrak{A}_n^m} (1 - \mathfrak{J}(\mathbf{c}, \mathbf{a})T)$$

where the product is taken over all twisted Jacobi sums, $\mathfrak{J}(\mathbf{c}, \mathbf{a})$.

Studying the asymptotic behaviour of the zeta-function as $s \to r$ clearly boils down, then, to studying the asymptotic behaviour of the polynomial $Q(\mathcal{V}, q^{-s})$ as s tends to r, $0 \leq r \leq n$. To do this, we first evaluate the polynomials $Q(\mathcal{V}_A, q^{-r})$ corresponding to motives $\mathcal{V}_A$ as $s \to r$, and then glue together the motivic quantities to yield the following global results.

THEOREM 0.7
Let $\mathcal{V} = \mathcal{V}_n^m(\mathbf{c})$ be a diagonal hypersurface with twist $\mathbf{c}$ and let $\mathcal{X} = \mathcal{V}_n^m(\mathbf{1})$ be the Fermat variety, both of dimension n and degree m over $k = \mathbb{F}_q$.

(I) *Let $n = 2d$ be even, and assume that the degree m is prime, and that $m > 3$. Put $Q^*(\mathcal{V}, T) = (1 - q^d T)Q(\mathcal{V}, T)$. Define quantities $\varepsilon_d(\mathcal{V}_k)$, $\delta_d(\mathcal{V}_k)$ and $w_{\mathcal{V}}(r)$, as follows:*

$$\varepsilon_d(\mathcal{V}_k) = \frac{\rho_d(\mathcal{V}_{\bar{k}}) - \rho_d(\mathcal{V}_k)}{m-1}, \quad \delta_d(\mathcal{V}_k) = \frac{B_n(\mathcal{V}) - \rho_d(\mathcal{V}_k)}{m-1},$$

and for any r, $0 \leq r \leq n$,

$$w_{\mathcal{V}}(r) = \sum_{i=0}^{r}(r - i)h^{i,n-i}(\mathcal{V}).$$

Then the following assertions hold for the limit

$$\lim_{s\to d}\frac{Q^*(\mathcal{V},q^{-s})}{(1-q^{d-s})^{\rho_d(\mathcal{V}_k)}}.$$

1. *If $\mathcal{V}$ is supersingular (resp. strongly supersingular), then the limit is equal to $\pm m^{\varepsilon_d(\mathcal{V}_k)}$ (resp. equal to 1).*
2. *If $\mathcal{V}$ is of Hodge–Witt type, then the limit takes the following form:*

$$\pm\frac{B^d(\mathcal{V}_k)m^{\delta_d(\mathcal{V}_k)}}{q^{w_{\mathcal{V}}(d)}}.$$

Here $B^d(\mathcal{V}_k)$ is the global "Brauer number" of $\mathcal{V}_k$. It is a positive integer, and is a square up to powers of m.

If $\mathbf{c}$ is extreme, then $B^d(\mathcal{V}_k)$ is a square.

(II) *Let $n=2d+1$ and $m>3$ be prime. Then for any integer r, $0\le r\le d$,*

$$Q(\mathcal{V},q^{-r})=\frac{D^r(\mathcal{V}_k)}{q^{w_{\mathcal{V}}(r)}}$$

where $D^r(\mathcal{V}_k)$ is a positive integer, and $D^r(\mathcal{V}_k)=D^{n-r}(\mathcal{V}_k)$.

Detailed accounts of Theorem 0.7 can be found in Chapters 6 and 7 below. The hypothesis of m being prime is not a subtle one, and is present mostly for technical reasons. One expects that there are similar (though perhaps more complicated) formulas for the cases of composite m. Our calculations are in general agreement with this expectation; see the comments in Chapter 9.

For diagonal hypersurfaces $\mathcal{V}=\mathcal{V}_2^m(\mathbf{c})$ of dimension $n=2$ and degree $m>3$ with twist $\mathbf{c}$ over $k=\mathbb{F}_q$, the Tate conjecture holds for $\mathcal{V}$ over k, so that $\mathcal{V}$ satisfies the Artin–Tate formula relative to k (cf. Milne [Mil75]). One of the motivations of the Lichtenbaum–Milne conjecture is to generalize the Artin–Tate formula to higher (even) dimensional varieties. For diagonal hypersurfaces $\mathcal{V}=\mathcal{V}_n^m(\mathbf{c})$ of dimension $n=2d$ with twist $\mathbf{c}$ over $k=\mathbb{F}_q$, Lichtenbaum and Milne have shown that assuming the existence of complexes of étale sheaves $\mathbb{Z}(r)$ having certain properties yields the following formula:

THEOREM 0.8
Assume the étale complexes $\mathbb{Z}(r)$ exist and satisfy the conditions in [Mil86, Mil88]. Let $\mathcal{V}=\mathcal{V}_n^m(\mathbf{c})$ be a diagonal hypersurface of dimension $n=2d$ and (prime) degree $m>3$ with twist $\mathbf{c}$ over $k=\mathbb{F}_q$. Assume that the cycle map $\mathrm{CH}^d(\mathcal{V}_k)\to H^n(\mathcal{V}_k,\mathbb{Z}(d))$ is surjective and that the Tate conjecture holds for $\mathcal{V}_k$. Then $\mathcal{V}_k$ satisfies the Lichtenbaum–Milne formula:

$$\lim_{s\to d}\frac{Q^*(\mathcal{V},q^{-s})}{(1-q^{d-s})^{\rho_d(\mathcal{V}_k)}}=\pm\frac{\#\,\mathrm{Br}^d(\mathcal{V}_k)|\det A^d(\mathcal{V}_k)|}{q^{\alpha_d\mathcal{V}(d)}},\qquad (\dagger)$$

where $\mathrm{Br}^d(\mathcal{V}_k) = \#H^{n+1}(\mathcal{V}_k, \mathbb{Z}(d))$ *is the "Brauer" group of* $\mathcal{V}_k$ *and* $\#\,\mathrm{Br}^d(\mathcal{V}_k)$ *is its order,* $A^d(\mathcal{V}_k)$ *is the image of the d-th Chow group* $\mathrm{CH}^d(\mathcal{V}_k)$ *in* $H^n(\mathcal{V}_{\bar{k}}, \hat{\mathbb{Z}}(d))$, $\{D_i \mid i = 1, \cdots, \rho_d(\mathcal{V}_k)\}$ *is a* $\mathbb{Z}$*-basis for* $A^d(\mathcal{V}_k)$, $\det A^d(\mathcal{V}_k) = \det(D_i \cdot D_j)$ *is the determinant of the intersection matrix on* $A^d(\mathcal{V}_k)$*, and* $\alpha_{\mathcal{V}}(d) = s^{n+1}(d) - 2s^n(d) + w_{\mathcal{V}}(d)$ *where* $w_{\mathcal{V}}(d) = \sum_{i=0}^{d}(d-i)h^{i,n-i}(\mathcal{V})$ *with* $h^{i,j} = \dim_k H^j(\mathcal{V}, \Omega^i)$*, and* $s^i(d) = \dim \underline{H}^i(\mathcal{V}, \mathbb{Z}_p(d))$ *(as a perfect group scheme).*

For the definition of $\underline{H}$, see Milne [Mil86], p. 307.)

We refer to the formula in this theorem as the Lichtenbaum–Milne formula. It is known to hold for $d = 1$ or $d = 2$ whenever the Tate conjecture holds. When the Brauer group $\mathrm{Br}^d(\mathcal{V}_k)$ exists, its order is a square, and this gives us a handle on the (otherwise quite mysterious) value of this term in the formula.

Since we can get information about the special values directly from properties of twisted Jacobi sums, we can compare these results with those predicted by the Lichtenbaum–Milne formula.

THEOREM 0.9
The notation of Theorem 0.8 remains in force. Assume that m is prime (so the Tate conjecture holds), that the complexes $\mathbb{Z}(r)$ *exist, and that the cycle map* $\mathrm{CH}^d(\mathcal{V}_k) \to H^n(\mathcal{V}_k, \mathbb{Z}(d))$ *is surjective (so the Lichtenbaum–Milne formula (†) on the preceding page is valid). Then we have, for m prime:*

(I) *The following assertions hold:*

1. *If* $\mathcal{V}_k$ *is supersingular, then*

$$\#\,\mathrm{Br}^d(\mathcal{V}_k)|\det A^d(\mathcal{V}_k)| = q^{\alpha_{\mathcal{V}}(d)} m^{\varepsilon_d(\mathcal{V}_k)}.$$

2. *If* $\mathcal{V}_k$ *is of Hodge–Witt type, then*

$$\#\,\mathrm{Br}^d(\mathcal{V}_k)|\det A^d(\mathcal{V}_k)| = B^d(\mathcal{V}_k) m^{\delta_d(\mathcal{V}_k)}.$$

(II) *For each prime* ℓ *with* $(\ell, m) = 1$*, the following assertions hold:*

1. *For a prime* ℓ *with* $(\ell, mp) = 1$*,*

$$\#\,\mathrm{Br}^d(\mathcal{V}_k)_{\ell-\mathrm{tors}} = \begin{cases} 1 & \text{if } \mathcal{V}_k \text{ is supersingular} \\ |B^d(\mathcal{V}_k)|_\ell^{-1} & \text{if } \mathcal{V}_k \text{ is of Hodge–Witt type} \end{cases}$$

and

$$|\det A^d(\mathcal{V}_k) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell| = 1.$$

2. *For the prime* $p = \mathrm{char}(k)$*, if* $\mathcal{V}_k$ *is of Hodge–Witt type, then*

$$\#\,\mathrm{Br}^d(\mathcal{V}_k)_{p-\mathrm{tors}} = |B^d(\mathcal{V}_k)|_p^{-1} \quad \text{and} \quad |\det A^d(\mathcal{V}_k) \otimes_{\mathbb{Z}} \mathbb{Z}_p| = 1.$$