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Constructivism in Mathematics

An Introduction

VOLUME I

A.S. TROELSTRA and D. van DALEN

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AN INTRODUCTION

VOLUME I

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CONSTRUCTIVISM IN MATHEMATICS: CONTENTS

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CONSTRUCTIVISM IN MATHEMATICS

VOLUME I

This book is dedicated
to the memory of our teacher
Arend Heyting

PREFACE

The present volume is intended as an all-round introduction to constructivism. Here constructivism is to be understood in the wide sense, and covers in particular Brouwer's intuitionism, Bishop's constructivism and A.A. Markov's constructive recursive mathematics. The ending "-ism" has ideological overtones: "constructive mathematics is the (only) right mathematics"; we hasten, however, to declare that we do not subscribe to this ideology, and that we do not intend to present our material on such a basis.

The first successful introduction to constructive mathematics, more specifically to intuitionistic mathematics, was Heyting's "Intuitionism, an introduction", which first appeared in 1956 and went through three editions. At the moment of its appearance Heyting's book was unique; now there is a whole collection of introductory texts and monographs dealing with constructive mathematics in its various forms, as well as with the metamathematics of constructive mathematics. Let us therefore indicate the principal features of the present book.

(a) It treats constructive mathematics in its various forms, as well as the metamathematics of constructivism. Each can serve as useful background information for the other.

(b) The treatment of constructive mathematics does not attempt to be systematic, only to give some illustrative examples.

(c) We have tried to select topics we expected to retain their interest for some time to come; we have not tried to follow the latest fashions.

(d) The treatment of the metamathematics is intended to be essentially self-contained; therefore we have included certain details, such as the formalization of elementary recursion theory, which are not part of standard introductory texts on mathematical logic. We assume only some familiarity with classical first-order predicate logic, preferably in natural deduction style, such as may be obtained from many introductory texts. Some familiarity with elementary recursion theory, though not absolutely necessary, is also helpful.

The material suitable for a first introduction to the topic is properly contained in the first six chapters. More specifically, the introductory material consists of chapter 1, sections 2.1–6, 3.1–5, 4.1–7 and the chapters 5 and 6. In addition the chapters 1–4 contain more specialized sections, which have been placed there for systematic reasons. We have tried to avoid too much interdependence in the introductory material; hence there are several selections for an introductory course possible, for example:

Selection 1, with emphasis on examples of constructive mathematics, consists of the following sections and subsections: 1.1–3, 3.3.1–2, 3.3.6 (first proof), 3.4, 4.1–3, 4.5, 5.1–4, 6.1–2, 4.6.1–10, 4.6.13–14, 4.7, 6.3, 6.4.1–11, 6.4.13, 4.4 (sketchy), 6.4.12.

The logical aspects may be given somewhat more emphasis by also treating 2.1, 2.3.1–6, 2.5.1–11, and the whole section 3.3 except for the second proof in 3.3.6.

Another possible extension of this selection is 5.5–7.

Selection 2, with emphasis on logical and metamathematical aspects, consists in the following sections and subsections: 1.1–3, 2.1, 2.3.1–6, 2.4.1–4 (optional), 2.5.1–13, 2.6 (optional), 3.1–5, 4.1–7, possibly extended with 5.1–4.

Occasional reference to material outside these selections may be skipped. These selections can be varied in many different ways.

The treatment of the introductory material is more leisurely and detailed than in subsections and later chapters of a more specialized nature; there we often leave the routine proofs as an exercise for the reader.

Most of the material has been tried out in courses at Amsterdam and Utrecht.

All chapters, except the last one, have a final section entitled "Notes". These are reserved for credits, historical remarks, asides, suggestions for further reading etcetera. The historical remarks deal primarily with the constructivist tradition and its metamathematics. We did not attempt to trace systematically the history of all the topics treated, and there is no pretense of completeness.

The final chapter of volume 2, the "Epilogue", contains some observations of a general nature prompted by the technical developments in the earlier chapters, as well as a brief discussion of some controversial issues.

In order to make independent use of the first volume possible, we have provided a separate index, bibliography and list of symbols to volume 1.

Due to circumstances beyond our control the burden of authorship has not been equally divided. Specifically, the first author produced the bulk of the material and the second author wrote section 1.4 and chapter 8. Chapter

3 is the joint responsibility of both authors. Furthermore both authors accept the responsibility for the contents in the sense that the material was the subject of mutual discussions, criticisms and emendations.

H. Luckhardt read through an early draft of the manuscript and provided a long list with corrections.

We owe a special debt to I. Moerdijk, who over the past few years rapidly reversed the roles in the teacher–pupil relationship. In chapters 13–15 we made at several places liberal use of handwritten notes prepared by him. André Hensbergen tested the exercises of chapters 7 and 10, Rineke Verbrugge assisted us with research for some of the notes.

The job of preparing the final version of the manuscript was made considerably easier by Yvonne Voorn, who typed a draft of the manuscript on a wordprocessor.

Muiderberg/De Meern
January 1987

A.S. TROELSTRA
D. VAN DALEN

PRELIMINARIES

1. *Internal references.* Within chapter n, "$k.m$" refers to the subsection numbered $k.m$ in chapter n; "section k" refers to section k of chapter n; "section $k.m$" refers to section m of chapter k ($k \neq n$), and "$k.l.m$" refers to subsection $l.m$ of chapter k ($k \neq n$). "Exercise $k.l.m$" or "E$k.l.m$" refers to the exercise numbered $k.l.m$ at the end of chapter k. An exercise numbered $k.l.m$ should be regarded as belonging to section $k.l$.

2. *Bibliographical references.* are given as author's name followed by year of publication, possibly followed by a letter in the case of more than one publication in the same year by the same author, e.g. "as shown by Brouwer (1923)...", or "... in Brouwer (1923A) it was proved that...", or "the bar theorem (Brouwer 1954)...". References to works by two authors appear as "Kleene and Vesley (1965)".

In the case of three or more authors we use the abbreviation "et al.", e.g. "Constable et al. (1986)". The bibliography at the end of the book contains only items which are actually referred to in the text; by the completion of the logic bibliography Müller (1987) has relieved us of the task of providing something approaching a complete bibliography of constructivism.

In the rest of this section some general notational conventions are brought together, to be consulted when needed. There may be local deviations from these conventions. At the end of the book there is an index of notations of more than purely local use.

3. *Definitions.* $\equiv$ indicates *literal identity*, modulo renaming bound variables. $:=$ is used as the *definition symbol*, the defining expression appears on the right hand side.

4. *Variables, substitution.* The concept of free and bound variable is defined as usual; for the sets of free and bound variables of the expression $\mathfrak{a}$ we use

FV($\mathfrak{a}$) and BV($\mathfrak{a}$) respectively. As a rule we regard expressions which differ only in the names of bound variables as isomorphic; that is to say, bound variables are used as position markers only.

For the result of *simultaneous substitution* of $t_1, \ldots, t_n$ for the variables $x_1, \ldots, x_n$ in the expression $\mathfrak{a}$ we write $\mathfrak{a}[x_1, \ldots, x_n/t_1, \ldots, t_n]$. We shall often use a looser notation: once $\mathfrak{a}(x_1, \ldots, x_n)$ has appeared in a context, we write $\mathfrak{a}(t_1, \ldots, t_n)$ for $\mathfrak{a}[x_1, \ldots, x_n/t_1, \ldots, t_n]$. Using vector notation as an abbreviation we can also write $\mathfrak{a}[\vec{x}/\vec{t}\,]$ for $\mathfrak{a}[x_1, \ldots, x_n/t_1, \ldots, t_n]$.

In using the substitution notation we shall as a rule tacitly assume the terms to be free for the variables in the expression considered (or we assume that a suitable renaming of bound variables is carried out). In our description of logic in chapter 2 we are more explicit about these matters.

We shall frequently economize on parentheses by writing Ax or At (A a formula) instead of $A(x)$ or $A(t)$.

5. *Logical symbols.* As logical symbols we use

$$\bot, \neg, \vee, \wedge, \rightarrow, \leftrightarrow, \forall, \exists, \exists!$$

both formally and informally. In bracketing we adopt the usual convention that $\neg, \forall, \exists$ bind stronger than any of the binary operators, and that $\wedge, \vee$ bind stronger than $\rightarrow, \leftrightarrow$. Occasionally dots are used as separating symbols instead of parentheses. In discussing formal systems it will be usually clear from the context whether the symbol is used as part of the formalism or on the metalevel. Where it is necessary to avoid confusion we use $\Rightarrow, \Leftrightarrow$ on the metalevel; sometimes a comma serves as a conjunction, and "iff" abbreviates "if and only if". Unless stated otherwise, $\neg, \leftrightarrow, \exists!$ are regarded as abbreviations defined by

$$\neg A := A \rightarrow \bot\ ;$$

$$A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A);$$

$$\exists! x A := \exists x (A \wedge \forall y (x = y \leftrightarrow A[x/y])).$$

For repeated equivalences we sometimes write $A \leftrightarrow B \leftrightarrow C \leftrightarrow \cdots$ meaning $(A \leftrightarrow B) \wedge (B \leftrightarrow C) \wedge \cdots$. For iterated finite conjunctions and disjunctions we use

$$\bigwedge, \bigvee.$$

Notation for restricted quantifiers:

$$\forall x \in X, \exists x \in X.$$

For iterated quantifiers we use the abbreviations

$$\forall x_1 x_2 \dots x_n := \forall x_1 \forall x_2 \dots \forall x_n, \quad \exists x_1 x_2 \dots x_n := \exists x_1 \exists x_2 \dots \exists x_n,$$

and for iterated restricted quantifiers

$$\forall x, y \in A := \forall x \in A \, \forall y \in A \text{ etcetera.}$$

For possibly undefined expressions α, $E\alpha$ means "α exists", or "α is well-defined" (cf. the use of $t\downarrow$ in recursion theory).

6. *Set-theoretic notation.* Our standard set-theoretic symbols are

$$\emptyset, \in, \notin, \subset, \supset, \setminus, \cap, \cup, \bigcap, \bigcup.$$

Here $\subset$ and $\supset$ are used for not necessarily proper *inclusion*, i.e. $X \subset Y := \forall x(x \in X \to x \in Y)$, etc. We also use the standard notations

$$\{x_1, x_2, \dots, x_n\} \quad \text{(for finite sets)},$$

$$\{x : A(x)\}, \{f(x) : A(x)\}, \{x \in B : A(x)\} \quad (f \text{ a function}).$$

For *complements* of sets relative to some fixed set we often use ^{c}. The fixed superset involved will be clear from the context. Thus if X is a subset of $\mathbb{N}$, we write X^c for $\mathbb{N} \setminus X$.

For finite *cartesian products* we use $\times$, for arbitrary cartesian products $\prod$.

If X is a set and $\sim$ an equivalence relation on X we write $X/\sim$ for the set of *equivalence classes* of X modulo $\sim$. For the *equivalence class* of an $x \in X$ we write $x/\sim$, $x_\sim$, $(x)_\sim$ or $[x]_\sim$.

The *set of functions* from X to Y is written as $X \to Y$ or Y^X. *Restrictions* are indicated by $\upharpoonright$ or $|$. $\mathrm{P}(X)$ is the *powerset* of X.

"f is a function from X to Y" is written as $f \in X \to Y$ (and only occasionally as $f : X \to Y$); the use of this notation must not be regarded as a commitment to the set of all functions from X to Y as a well-defined totality. If t is a term we can also introduce "t regarded as a function of the parameter (variable) x" by one of the following notations:

$$\lambda x.t \quad \text{or} \quad f : x \mapsto t.$$

Notations (in diagrams) for *injections*, *surjections*, *bijections* and *embeddings* are $\rightarrowtail$, $\twoheadrightarrow$, $\mapsto\!\!\!\!\twoheadrightarrow$, $\hookrightarrow$ respectively.

For the *characteristic function* of a relation R we use χ_R, where $\chi_R(t) = 0 \leftrightarrow R(t)$.

For the function f applied to the argument t we usually write $f(t)$, or even ft, if no confusion can arise; in general we drop parentheses whenever we can safely do so. In certain chapters we use $t(t')$ or tt' for term t applied to term t' (e.g. in chapter 9); in such cases we use square brackets instead of parentheses to refer to occurrences in a term, writing $t[x]$ instead of $t(x)$ etc.

Pairs, or *n-tuples* for fixed length, are usually also indicated by means of parentheses (,); in the case of finite sequences of variable length we use $\langle\ ,\ \rangle$: $\langle x_1, x_2, \ldots, x_n\rangle$ is a sequence of length n (notation $\mathrm{lth}\langle x_1, x_2, \ldots, x_n\rangle = n$). For concatenation we use $*$. A sequence of arguments may also be indicated by vector notation; thus we write $f(\vec{t})$ or $f\vec{t}$ for $f(t_1, \ldots, t_n)$, where $\vec{t} \equiv (t_1, \ldots, t_n)$; $\vec{t} = \vec{s}$ indicates equality of vectors.

We use the abbreviation $\lambda x_1 x_2 \ldots x_n.t$ for $\lambda x_1(\lambda x_2 \ldots(\lambda x_n.t)\ldots)$. If we wish to regard t as a function in n arguments $x_1, x_2, \ldots, x_n$ we use comma's: $\lambda x_1, x_2, \ldots, x_n.t$.

For the *graph* of the function f we write graph(f), dom(f) is its *domain*, range(f) its *range*.

7. *Mathematical constants.* The following constants

$$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{B}, \mathbb{C}$$

denote the natural numbers, the integers, the rationals, the reals, Baire space and the complex numbers respectively. Throughout the book, n, m, i, j, k, unless indicated otherwise, are supposed to range over $\mathbb{N}$, and $\alpha, \beta, \gamma, \delta$ over $\mathbb{B}$ or a subtree of $\mathbb{B}$. In metamathematical work we use $\bar{n}, \bar{m}$ for numerals.

For an infinite sequence $\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2, \ldots$ (i.e. a function with domain $\mathbb{N}$) we use the notation $\langle \mathfrak{a}_n \rangle_n$. A notation such as $\lim\langle x_n\rangle_n$ is self-explanatory.

The notation for arithmetical operations is standard; the multiplication dot $\cdot$ is often omitted.

8. *Formal systems and axioms.* Formal systems are designated by combinations of roman boldface capitals, e.g. **HA**, **IQC**, $\mathbf{EM}_0\!\upharpoonright$, etc.

If **H** is a system based on intuitionistic logic, $\mathbf{H}^c$ is used for the corresponding system based on classical logic.

For the language of a formal system **H** we often write $\mathscr{L}(\mathbf{H})$. If $\mathbf{H}'$ extends **H**, and $\mathscr{L}(\mathbf{H}')$ extends $\mathscr{L}(\mathbf{H})$, while $\mathbf{H}' \cap \mathscr{L}(\mathbf{H}) = \mathbf{H}$, then $\mathbf{H}'$ is said to be a *conservative extension* of **H** ($\mathbf{H}'$ is *conservative over* **H**). A *definitional extension* is a special case of a conservative extension, where the extra symbols of $\mathscr{L}(\mathbf{H}')$ can be replaced by explicit definitions (cf. 2.7.1).

Axiom schemas and rules are usually designated by combinations of roman capitals: REFL, BI, BI_{D}, FAN, WC-N, CT_0 etc.

$\mathbf{H} \vdash A$ means "A is derivable in the system **H**", and $\mathbf{H} + \mathrm{XYZ} \vdash A$ "A is derivable in **H** with the axiom schema or rule XYZ added". Occasionally we use subscript notation: $\vdash_{\mathbf{H}} A$ instead of $\mathbf{H} \vdash A$.

9. *Validity.* For validity of a sentence A in a model $\mathscr{M}$ we use the notation $\mathscr{M} \models A$. It is to be noted that if the relations, functions and constants of the model $\mathscr{M}$ are defined constructively, then $\mathscr{M} \models A$ also makes sense constructively (cf. 2.5.1).

CONTENTS OF VOLUME I

CHAPTER 1

INTRODUCTION

In this chapter we first present a brief characterization of the various forms of constructivism which shall play a role in the book later on. After the review of constructivist trends, we illustrate the idea of constructivity by means of some simple and quite elementary examples in section 2; most of the points discussed will receive a more thorough treatment in later chapters.

Section 3 introduces the Brouwer–Heyting–Kolmogorov interpretation of the logical operators, and the method of weak counterexamples.

Section 4 gives a brief historical survey.

1. Constructivism

1.1. Time and again, over the last hundred years certain mathematicians have defended an approach to mathematics which might be called "constructive" in the broad sense used in this book, in more or less explicit opposition to certain forms of mathematical reasoning used by the majority of their colleagues. Some of these critics of the mathematics of their (our) time not only criticized contemporary mathematical practice, but actually endeavoured to show how mathematics could be rebuilt on constructivist principles.

There are, however, considerable differences in outlook between the various representatives of constructivism; constructivism in the broad sense is by no means homogeneous, and even the views expressed by different representatives of one "school", or by a single mathematician at different times are not always homogeneous. Our descriptions below present a simplified picture of this complex reality. We shall attempt a brief characterization of the principal constructivist "trends" or "schools" playing a role in this book.

In our discussions we shall use the adjective "classical" for logical and mathematical reasoning based on the usual two-valued logic, in which every meaningful statement is assumed to be either true or false.