

Calculus

WITH ANALYTIC GEOMETRY AND LINEAR ALGEBRA

Leopoldo V. Toralballa

DEPARTMENT OF MATHEMATICS
NEW YORK UNIVERSITY
BRONX, NEW YORK



ACADEMIC PRESS

NEW YORK AND LONDON

TO THE MEMORY OF
ENRIQUE, CONSOLACION, AND ASUNCION
THIS BOOK IS AFFECTIONATELY DEDICATED

COPYRIGHT © 1967 BY ACADEMIC PRESS INC.

All rights reserved.

No part of this book may be reproduced in any form,
by photostat, microfilm, or any other means, without
written permission from the publishers.

ACADEMIC PRESS INC.

111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by

ACADEMIC PRESS INC. (LONDON) LTD.

Berkeley Square House, London, W.1

Library of Congress Catalog Card Number: 65-26412

PRINTED IN THE UNITED STATES OF AMERICA

Calculus

WITH ANALYTIC GEOMETRY AND LINEAR ALGEBRA

Academic Press
Textbooks in Mathematics

EDITORIAL BOARD:

Ralph P. Boas, Jr., *Northwestern University*
Herman Gluck, *Harvard University*

GEORGE BACHMAN and LAWRENCE NARICI. Functional Analysis

P. R. MASANI, R. C. PATEL, and D. J. PATIL. Elementary Calculus

WILLIAM PERVIN. Foundations of General Topology

ALBERT L. RABENSTEIN. Introduction to Ordinary Differential Equations

JAMES SINGER. Elements of Numerical Analysis

EDUARD L. STIEFEL. An Introduction to Numerical Mathematics

HOWARD G. TUCKER. An Introduction to Probability and Mathematical Statistics

CHIH-HAN SAH. Abstract Algebra

LEOPOLDO V. TORALBALLA. Calculus with Analytic Geometry and Linear Algebra

In preparation

DONALD W. BLACKETT. Elementary Topology: A Combinatorial and Algebraic Approach

Preface

This book presents the material for a first course in calculus and analytic geometry, together with an introduction to linear algebra, and is optimally covered in three semesters. It is written on a level somewhat above that found in the average text on elementary calculus, for much valuable time can be saved if the student's first course in calculus is sufficiently rigorous so that no serious backtracking will be necessary when he studies advanced calculus. In this connection, the current Committee on the Undergraduate Program in Mathematics says pointedly: "The basic concepts should be introduced in the same spirit in which they are used by working mathematicians, and proofs ought to have the same clarity and elegance which distinguishes all first rate mathematics." I have tried to keep these views constantly in mind while writing this book.

But rigor alone is not enough! For what is rigor in mathematics other than accuracy and precision in the formulation of new definitions and concepts, and correctness in the proof of theorems? The formal presentation of each new concept in this book is always preceded by an intuitive discussion, and the formal proofs are generally preceded by heuristic ones, much in the fashion in which one lectures.

This text was planned to be used along two different tracks, according to the aptitudes and interests of the student and the breadth of the curriculum. Track I is the principal track for this book, and corresponds to the standard introductory calculus course taken by the majority of freshmen and sophomores in our colleges and universities today. It is intended for those students who need a thorough familiarity with the calculus as a scientific tool, and for those mathematics majors who do not follow an honors program. The student is led to an intimate acquaintance with all the leading concepts and ideas of the calculus. Proofs are given for all except a few of the most abstract theorems, and the course on this level is designed for

most students majoring in physics, chemistry, or any of the engineering disciplines, as well as for a portion of those majoring in mathematics.

Track II is the honors course. It is designed for those students who seek an early mastery of the concepts and ideas of the calculus as a stage toward more advanced work in mathematics and related fields. Here, ideas are presented in their modern form and proofs are given in detail. This course will meet the needs of the more talented students majoring in mathematics, physics, and chemistry, as well as those students who have had some advanced preparation in calculus on the high school level.

The complete description of the material covered by Track I is given in a table to be found immediately following this preface. It is assumed that students following Track II will cover the entire volume.

This double track arrangement incorporates a number of significant advantages:

(1) It facilitates the switching back and forth of students who have originally been placed in courses above or below their own level. If the same text is used on the two different tracks between which the student is switching, the transfer invariably goes along much more smoothly than otherwise.

(2) It provides the students on Track I with built-in "outside reading." It is folly of course to imagine that this arrangement can serve as a total replacement for outside reading, but it does give the more eager or more talented student an opportunity to seek out advanced ideas or detailed proofs which are already presented in the familiar language and spirit of his own text.

(3) There is an advantage for the honors student, who is traveling along Track II. For even on this track it is the rare student who can digest every new idea in its most sophisticated form without the softening cushion of a relaxed intuitive introduction, as is provided in this book.

After a brief introduction to set theory, the book begins with a postulational presentation of the real number system. As is well known, the theorems of the calculus are all rooted in the properties of the real number system, and it is thus impossible to give a rigorous treatment of the former without first giving an appropriate treatment of the latter. Of the various approaches to the real number system, the postulational presentation seems to me to be pedagogically superior to all the others for use in a first course in calculus. After the absolute value of a real number is defined, the concept of a neighborhood of a real number is introduced. This is followed by a well-motivated, simple, yet logically adequate development of the topology on the space of real numbers.

Since the central theorem of the calculus is embodied in the statement concerning the relation between the derivative and the integral, these two concepts are presented together. The derivative is presented historically, as an outgrowth of the effort to solve the problem of tangents and the problem of instantaneous velocities. The integral, in turn, is presented as an outgrowth of the method of exhaustions. The existence of the integral of a continuous function is proved.

A chapter on linear algebra has been included, partly because the discussion of functions of several variables makes use of some results about linear systems and quadratic forms, and partly because the modern abstract theory of vector spaces serves so well to illuminate the subject of vectors in Euclidean spaces. A thoroughgoing vectorial presentation is made of the analytic geometry of 3-dimensional Euclidean space.

The customary treatments of such geometric concepts as area, volume, centroid, and moment of inertia make no attempt to show that these concepts are independent of the particular frame of reference and of the mode of decomposition. I feel that such an omission is out of consonance with the modern view in mathematics, and have on this account included in the chapter on multiple integrals a detailed though elementary discussion of Jordan content.

The subject of surface area is one that is almost invariably treated very superficially in texts on the calculus, both on the elementary and advanced levels. An impression is thereby unwittingly created that the subject is so profound that it were best for the student to be satisfied with the rather arbitrary treatment that is being given. I feel that this is both unfortunate and unnecessary, and in the chapter on multiple integrals I have presented the subject of surface area in a manner that I believe is more natural and more satisfying.

While a high level of rigor is maintained throughout the text, I have generally preferred the simple and direct approach to the sophisticated one when the latter is neither necessary nor particularly convenient.

It is my earnest hope that this book will be of help to all those who seek a mastery of the basic concepts of the calculus.

I am grateful to my wife Gloria and to my son Lee for the encouragement they gave me during the writing of this book. To Mrs. Sadelle Wladaver, I wish to express my thanks for her fine job of typing the manuscript. I have benefited from discussions which I had with Professor Fred Ficken, chairman of the mathematics department at New York University, and with many of my colleagues at New York University, especially, with Professor Hilbert Levitz. I wish to express my deep feeling of gratitude to Professor Herman Gluck of Harvard University and to Professor Paul Sally of the University of Chicago. Professor Gluck suggested many improvements in the presentation of the material. Professor Sally read thoroughly and critically the entire manuscript, pointed out many incongruities, and suggested some very valuable emendations. I am very grateful to the staff of Academic Press for their unfailing cooperation.

L. V. TORALBALLA

New York, New York

Chapter	Track I
Introduction	Secs. 0.1, 0.2, 0.4, 0.5
1	Introd., Secs. 1.1–1.6 (omit pf. of B-W Thm.), 1.7, 1.8 (omit pfs.)
2	Introd., Secs. 2.1 (Omit pf. of Thm. 2.1), 2.2 (omit pfs. of Thms. 2.2–2.4), 2.3–2.9 (omit all pfs. in Sec. 2.9), 2.10
3	Introd., Secs. 3.1–3.5, 3.8, 3.9 (omit pf. of Thm.3.6)
4	All
5	All
6	All
7	All
8	All (omit pf. of L'Hôpital's Thm.)
9	All (omit pfs. of Thms.)
10	All
11	Introd., Secs. 11.1–11.4
12	Introd., Secs. 12.2–12.4, 12.7
13	Secs. 13.1–13.4, 13.7, 13.9–13.11, 13.13, 13.14
14	Introd., Secs. 14.1–14.12
15	All
16	Secs. 16.1–16.8, 16.11, 16.13, 16.20, 16.21
17	Introd., Secs. 17.1–17.3 (omit Thms. 17.4, 17.5), 17.5 (omit Thm. 17.9), 17.8–17.10
18	Introd., Secs. 18.1–18.7

Contents

Preface. v

Introduction. Basic Concepts of Set Theory

- 0.1. Propositions, propositional functions, sets, 1
- 0.2. Ordered pairs, Cartesian products, 4
- 0.3. Binary relations, 5
- 0.4. Functions, 7
- 0.5. Operations, 9
- 0.6. Identity functions, inverse functions, 10
- 0.7. Similarity of two sets, cardinality, 10
- 0.8. Finite sets, infinite sets, 11
- 0.9. Ordered sets, 11

1 *The Real Number System*

- Introduction, 14
- 1.1. Definition of the real number system, 18
- 1.2. The integers, 21
- 1.3. The rational numbers, 25
- 1.4. Other properties of the real number system, 28
- 1.5. The distance between two real numbers, 29
- 1.6. Limit points and closed and open sets, 30
- 1.7. Sequences, 33
- 1.8. An extension of terminology, 37
- 1.9. Monotonic sequences, 37
- 1.10. Combinations of sequences, 39
- 1.11. Connectedness, 43

2 *Functions, Mappings, and Graphs*

- Introduction, 48
- 2.1. Continuity, 49
- 2.2. Attainment of LUB, GLB, and intermediate values, 53
- 2.3. Graphs, 54

- 2.4. Limits of functions, 57
- 2.5. Roots, 64
- 2.6. Irrational numbers, 65
- 2.7. Limits of certain radical functions, 65
- 2.8. An extension of limit terminology, 67
- 2.9. Uniform continuity, 70
- 2.10. Linear functions, 75

3 *The Derivative and the Integral*

- 3.1. The derivative, 77
- 3.2. The Lipschitz property, 86
- 3.3. Maxima and minima, 87
- 3.4. The law of the mean, 93
- 3.5. Differentiability and continuity, 96
- 3.6. Functions of bounded variation, 98
- 3.7. The uniform Lipschitz property, 99
- 3.8. A primitive of a function, 100
- 3.9. The integral, 100

4 *Introduction to the Applications of the Integral*

- Introduction, 122
- 4.1. Areas, 123
- 4.2. Centroids of plane regions, 127
- 4.3. Volumes, 134
- 4.4. Disk elements, 135
- 4.5. Shell elements, 137
- 4.6. Sectional elements, 140
- 4.7. Centroids of regions of revolution, 144
- 4.8. Work, 149
- 4.9. Hydrostatic force, 152

5 *The Derivative. Introduction to the Applications of the Derivative*

- 5.1. The derivative of the product of two functions, 155
- 5.2. The derivative of the quotient of two functions, 156
- 5.3. The chain rule, 156
- 5.4. The derivative of a power of a function, 157
- 5.5. Implicit functions, 160
- 5.6. Continuity of $F(x, y)$ at (x_0, y_0) , 161
- 5.7. The partial derivative, 161
- 5.8. The derivative of an implicit function, 162
- 5.9. Higher derivatives, 162
- 5.10. Taylor's expansion, 163
- 5.11. Graph tracing, 168
- 5.12. Critical points and points of inflection, 169
- 5.13. Symmetry, 169
- 5.14. Graphs of the rational functions, 172
- 5.15. Loci of equations of the form $y^2 = f(x)$, 177
- 5.16. Maxima and minima, 182
- 5.17. Time rates, 188
- 5.18. The differential, 193
- 5.19. The differential as a function of two variables, 195
- 5.20. Approximations, 197

6 Plane Analytic Geometry

- Introduction, 202
- 6.1. The distance between two points, 204
- 6.2. The coordinates of points of division, 204
- 6.3. Loci, 207
- 6.4. The straight line, 212
- 6.5. The slope, 215
- 6.6. The distance between a point and a line, 216
- 6.7. Families of straight lines, 220
- 6.8. Tangents and normals, 225
- 6.9. Symmetry, 228
- 6.10. The circle, 229
- 6.11. Pencils of circles, 231
- 6.12. The conics, 235
- 6.13. The parabola, 238
- 6.14. Geometric properties of the parabola, 241
- 6.15. The ellipse, 244
- 6.16. Geometric properties of the ellipse, 247
- 6.17. The hyperbola, 252
- 6.18. Geometric properties of the hyperbola, 254
- 6.19. Coordinate transformations, 261
- 6.20. Translation of axes, 262
- 6.21. Rotation of axes, 263
- 6.22. Coordinate systems, 265
- 6.23. Curve tracing, 268

7 Elementary Functions

- 7.1. The logarithmic function $\ln x$, 273
- 7.2. An alternative presentation of the logarithmic function, 277
- 7.3. The exponential function e^x , 278
- 7.4. The exponential function a^x , $a > 0$, 280
- 7.5. The logarithmic function $\log_b x$, where $b > 0$, 281
- 7.6. The power function x^p , where $x > 0$ and p is any real number, 282
- 7.7. Logarithmic differentiation, 283
- 7.8. Roots, 284
- 7.9. The trigonometric functions, 285
- 7.10. Other inverse trigonometric functions, 295
- 7.11. The hyperbolic functions, 299
- 7.12. Inverse hyperbolic functions, 302
- 7.13. Parametric equations, 306
- 7.14. Geometric representation of the x in the trigonometric functions, 310

8 Applications of the Derivative

- 8.1. Tangents and normals, 312
- 8.2. Maxima and minima, 316
- 8.3. Time rates, 320
- 8.4. Differential equations, 322
- 8.5. Application of the differential equation $dy/dx = ky$, 322
- 8.6. Differential equations of the form $d^n y/dx^n = f(x)$, 323
- 8.7. Approximate formulas, 325
- 8.8. Evaluation of certain indeterminate forms, 328
- 8.9. Linear motion, 338
- 8.10. Solution of equations in one unknown, 342

9 *The Search for Primitives. Applications of the Integral*

- 9.1. The systematic search for primitives, 348
- 9.2. Algebraic substitution, 355
- 9.3. Trigonometric substitution, 356
- 9.4. Integration by parts, 362
- 9.5. Decomposition into partial fractions, 365
- 9.6. Some special types of integrals, 371
- 9.7. Improper integrals of the first kind, 374
- 9.8. Improper integrals of the second kind, 382

10 *Further Applications of the Integral*

- 10.1. The length of a curve, 388
- 10.2. Curves, 389
- 10.3. The identification of θ in $\sin \theta$, $\cos \theta$, and $\tan \theta$ with the angle θ , 397
- 10.4. The centroid of a curve, 398
- 10.5. The moment of inertia of a curve, 401
- 10.6. The area of a surface of revolution, 404
- 10.7. The curvature of a curve at a given point, 407
- 10.8. Applications of the integral in the polar coordinate system, 410
- 10.9. Approximate integration. Simpson's method, 417

11 *Infinite Series of Constants*

Introduction, 424

- 11.1. The geometric series, 425
- 11.2. The Cauchy condition for convergence, 426
- 11.3. Tests for convergence and divergence, 428
- 11.4. Operations with series, 442
- 11.5. Approximations to the sum of a convergent series, 445
- 11.6. The decimal representation of the real numbers, 451
- 11.7. The uniqueness of the real number system in the sense of isomorphism, 453

12 *Infinite series of Functions. Power Series*

- 12.1. Convergence, 457
- 12.2. Continuity, differentiability, and integrability of the sum of a series, 460
- 12.3. The domain of convergence, 464
- 12.4. The radius of convergence, 466
- 12.5. Properties of power series, 470
- 12.6. Representation of functions by power series, 471
- 12.7. The differential equation $\frac{dy}{dx} = ky$, k a constant, 480

13 *Linear Algebra*

- 13.1. The concept of group, 484
- 13.2. The concept of vector space, 485
- 13.3. Bases, 487
- 13.4. Linear functions, 489
- 13.5. Inner products, 490
- 13.6. The angle between two nonzero vectors, 491
- 13.7. Isomorphism, 495
- 13.8. The vector space $S_n(F)$, 496
- 13.9. Linear transformations from S^n to S^m . Matrices, 497
- 13.10. Square matrices, 499
- 13.11. Operations with matrices, 500
- 13.12. Determinants, 501

- 13.13. Expansion of determinants, 506
- 13.14. Solution of system of linear equations by determinants, 510
- 13.15. Square matrices, 516
- 13.16. Transformations, 517
- 13.17. Quadratic forms, 524

14 *The Euclidean Vector Spaces*

- Introduction, 527
- 14.1. Functions from E^1 to E^n , 528
- 14.2. Protovectors, 530
- 14.3. Directed segments, 532
- 14.4. Vectors, 535
- 14.5. Angle between two vectors, 537
- 14.6. Applications of vectors in elementary geometry, 538
- 14.7. An isomorphism, 541
- 14.8. The scalar product of two vectors, 542
- 14.9. A basis for E_n , 544
- 14.10. The vector space E_2 , 548
- 14.11. The vector space E_3 , 552
- 14.12. The coordinate components of a vector, 553
- 14.13. Vector functions of a real variable, 556
- 14.14. Planar motion, 558
- 14.15. The vector product of two vectors, 567
- 14.16. The scalar triple product, 571
- 14.17. The vector triple product, 572
- 14.18. Vector derivative formulas, 573

15 *Analytic Geometry of E^3*

- Introduction, 575
- 15.1. Rays. Straight lines, 576
- 15.2. Direction numbers, 577
- 15.3. Parallelism and orthogonality of lines, 578
- 15.4. The plane, 580
- 15.5. Orthogonality between a line and plane, 582
- 15.6. Parallelism and orthogonality of planes, 587
- 15.7. The normal form of the equation of a plane, 590
- 15.8. Surfaces, 594
- 15.9. The sphere, 595
- 15.10. The ellipsoid, 596
- 15.11. The hyperboloid of one sheet, 596
- 15.12. The hyperboloid of two sheets, 597
- 15.13. The elliptic paraboloid, 598
- 15.14. The hyperbolic hyperboloid, 599
- 15.15. Surfaces of revolution, 600
- 15.16. Cylinders, 601
- 15.17. Cones, 602
- 15.18. Curves and arcs in E^3 , 604
- 15.19. Tangents to an arc in E^3 , 608
- 15.20. Normal plane to an arc at a given point, 611
- 15.21. Length of arcs in E^3 , 611
- 15.22. Motion in E^3 of a particle, 616
- 15.23. Curvature, principal normal, and binormal, 616
- 15.24. The acceleration vector, 617
- 15.25. The cylindrical coordinate system, 619
- 15.26. The spherical coordinate system, 620

16 *Functions of Two or More Real Variables*

- Introduction, 622
- 16.1. Neighborhoods, 622
- 16.2. Sequences, 625
- 16.3. Continuity, 625
- 16.4. The partial derivative, 627
- 16.5. The law of the mean. First form, 630
- 16.6. The principal part of the increment of a function, 631
- 16.7. The differential, 636
- 16.8. The chain rule, 641
- 16.9. The uniform Lipschitz property, 644
- 16.10. The law of the mean. Symmetric form, 645
- 16.11. Time rates, 648
- 16.12. The partial derivatives of higher order, 649
- 16.13. An application, 651
- 16.14. The higher differentials, 654
- 16.15. Taylor's expansion, 656
- 16.16. The directional derivative, 660
- 16.17. The gradient, 664
- 16.18. The directional derivative of a function of three variables, 666
- 16.19. The gradient of a function of three variables, 668
- 16.20. Tangent planes and normal lines, 670
- 16.21. Maxima and minima, 673
- 16.22. The general case, 681
- 16.23. Implicit functions, 683
- 16.24. Implicit functions defined by several equations, 689
- 16.25. Transformations or mappings, 699
- 16.26. Products of transformations, 702
- 16.27. Functional dependence, 704
- 16.28. Constrained maxima and minima, 712

17 *Multiple Integrals*

- Introduction, 720
- 17.1. Intervals, 720
- 17.2. Limit points, 721
- 17.3. Continuity, 725
- 17.4. Areas and volumes, 728
- 17.5. Multiple integrals, 736
- 17.6. Upper sums and lower sums, 738
- 17.7. Multiple integrals on arbitrary closed, bounded, and connected sets, 744
- 17.8. Iterated integrals, 752
- 17.9. Changing the order of integration, 755
- 17.10. Multiple integrals as iterated integrals, 758
- 17.11. Change of variables, 766
- 17.12. Continuously differentiable transformations, 767
- 17.13. Differentiation under the integral sign, 779
- 17.14. Improper multiple integrals, 785

18 *Applications of the Multiple Integral*

- 18.1. Sets in E^2 . Areas, 792
- 18.2. Centroids, 798
- 18.3. Moments of inertia, 803
- 18.4. Moments of inertia with respect to a point, 808

- 18.5. Planar sets with mass, 809
- 18.6. Sets in E^3 . Volumes, 814
- 18.7. Centroids, 825
- 18.8. Moments of inertia, 833
- 18.9. Moments of inertia with respect to a line, 840
- 18.10. Sets with mass, 842
- 18.11. Area of a surface, 844
- 18.12. Improper integrals, 865
- 18.13. Area of a composite surface, 867
- 18.14. Surface area in cylindrical coordinates, 871
- 18.15. Surfaces of revolution, 872

Appendix A, 875

Appendix B, 877

Answers to Exercises, Set B, 882

Subject Index, 917

Introduction. Basic Concepts of Set Theory

0.1. *Propositions, propositional functions, sets*

A *proposition* is a statement or sentence of which it can be asked meaningfully whether the statement or sentence is true or false. For instance, the sentences

Kennedy was president of the United States
5 is greater than 10

are propositions. On the other hand the sentences

Virtue is green
A circle is honest

are not propositions.

A *propositional function* is a sentence which contains one and only one *variable*. For instance

x was president of the United States
 x is greater than 10

are propositional functions. They are not propositions. One sees, however, that if *Nixon* is substituted for x in the first sentence or *15* is substituted in the second sentence, one obtains a proposition.

Now, one finds it very convenient to associate with every given propositional function a *class*, the class of the objects, each of which, when substituted for the variable in the propositional function, yields a proposition that is true. Thus, to the propositional function

x is a friend of Richard

we associate the class of people who are friends of Richard. If John is a friend of Richard then we say that John is a member or an element of this class.