Optimization of Linear Control Systems

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Analytical Methods and Computational Algorithms

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GORDON AND BREACH SCIENCE PUBLISHERS
Australia • Canada • China • France • Germany • India
Japan • Luxembourg • Malaysia • The Netherlands • Russia
Singapore • Switzerland

Amsteldijk 166
1st Floor
1079 LH Amsterdam
The Netherlands

British Library Cataloguing in Publication Data

Aliev, F.A.
Optimization of linear control systems. — (Stability and control: theory, methods & applications; v. 8)
1. Linear control systems
I. Title II. Larin, Veniamin
003.7'4

ISBN 90-5699-113-2

Optimization of Linear Control Systems

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Introduction to the Series

The problems of modern society are both complex and interdisciplinary. Despite the apparent diversity of problems, tools developed in one context are often adaptable to an entirely different situation. For example, consider the Lyapunov's well known second method. This interesting and fruitful technique has gained increasing significance and has given a decisive impetus for modern development of the stability theory of differential equations. A manifest advantage of this method is that it does not demand the knowledge of solutions and therefore has great power in application. It is now well recognized that the concept of Lyapunov-like functions and the theory of differential and integral inequalities can be utilized to investigate qualitative and quantitative properties of nonlinear dynamic systems. Lyapunov-like functions serve as vehicles to transform the given complicated dynamic systems into a relatively simpler system and therefore it is sufficient to study the properties of this simpler dynamic system. It is also being realized that the same versatile tools can be adapted to discuss entirely different nonlinear systems, and that other tools, such as the variation of parameters and the method of upper and lower solutions provide equally effective methods to deal with problems of a similar nature. Moreover, interesting new ideas have been introduced which would seem to hold great potential.

Control theory, on the other hand, is that branch of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. To control an object implies the influence of its behavior so as to accomplish a desired goal. In order to implement this influence, practitioners build devices that incorporate various mathematical techniques. The study of these devices and their interaction with the object being controlled is the subject of control theory. There have been, roughly speaking, two main lines of work in control theory which are complementary. One is based on the idea that a good model of the object to be controlled is available and that we wish to optimize its behavior, and the other is based on the constraints imposed by uncertainty about the model in which the object operates. The control tool in the latter is the use of feedback in order to correct for deviations from the desired behavior. Mathematically, stability theory, dynamic systems and functional analysis have had a strong influence on this approach.

Volume 1, *Theory of Integro-Differential Equations*, is a joint contribution by V. Lakshmikantham (USA) and M. Rama Mohana Rao (India).

Volume 2, *Stability Analysis: Nonlinear Mechanics Equations*, is by A.A. Martynyuk (Ukraine).

Volume 3, *Stability of Motion of Nonautonomous Systems: The Method of Limiting Equations*, is a collaborative work by J. Kato (Japan), A.A. Martynyuk (Ukraine) and A.A. Shestakov (Russia).

Volume 4, *Control Theory and its Applications*, is by E.O. Roxin (USA).

Volume 5, *Advances in Nonlinear Dynamics*, is edited by S. Sivasundaram (USA) and A.A. Martynyuk (Ukraine) and is a multiauthor volume dedicated to Professor S. Leela.

Volume 6, *Solving Differential Problems by Multistep Initial and Boundary Value Methods*, is a joint contribution by L. Brugnano (Italy) and D. Trigiante (Italy).

Volume 7, *Dynamics of Machines with Variable Mass*, is by L. Cveticanin (Yugoslavia).

Volume 8, *Optimization of Linear Control Systems: Analytical Methods and Computational Algorithms*, is a joint work by F.A. Aliev (Azerbaijan) and V.B. Larin (Ukraine).

Due to the increased interdependency and cooperation among the mathematical sciences across the traditional boundaries, and the accomplishments thus far achieved in the areas of stability and control, there is every reason to believe that many breakthroughs await us, offering existing prospects for these versatile techniques to advance further. It is in this spirit that we see the importance of the 'Stability and Control' series, and we are immensely thankful to Gordon and Breach Science Publishers for their interest and cooperation in publishing this series.

Preface

This monograph describes new methods of analysis and synthesis of multivariable control systems together with the associated computer algorithms.

The analytical methods of synthesis of optimal linear stationary and periodic controlled systems, which generalize and unite the traditional approaches (frequency domain technique and state space method) are introduced. These methods allow us to obtain the efficient computation algorithms of synthesis optimal regulator and filter. The frequency domain method involving Wiener-Hopf equation (H_2-optimization) is based on an original parametrization procedure of the stabilizing regulator set and the particular cases of this approach are the Youla-Jabr-Bongiorno parametrization and Desoer-Liu-Murray-Saeks parametrization. We include some ingenious computing algorithms of solutions of Lyapunov and Riccati equation and generalized versions, and present new methods for spectral and J-spectral factorization of matrix polynomials and rational matrices and for calculation of the projections of the element of space L_2 on the space H_2. These are usually used as computational procedures in linear quadratic Gaussian (LQG) problem, and H_2 and H_∞-optimization. The algorithms applicability is illustrated by examples, published in the various journals and monographs (see for example *IEEE Trans. on Aut. Contr.* V. 35, 1990; V. 39, 1994; *Systems & Control Letters* V. 3, 1987; *Computer-Aided Control Systems Engineering*, North-Holland, 1985; Kucera V.A. *Discrete Linear Control: the polynomial equation approach*. Praha, Academia 1979; Petkov P. *et al. Computational methods for Linear Control Systems*, Prentice-Hall Inc., Eng. Cliffs, NJ, 1991) in numerical methods of linear controlled systems optimization.

The book is intended for students, post-graduates and engineers specializing in control systems and applied mathematics. It is presumed that the reader is acquainted with the fundamentals of linear theory of controlled systems, matrix algebra and has some experience in computing.

Contents

Introduction

One of the basic problems of control theory is the development of mathematical methods of synthesis of optimal systems functioning according to the principle of feedback. Impressive results in this direction have been obtained in the framework of the so-called linear quadratic Gaussian (LQG) problem (see Special Issue [1]), when an optimal regulator of the feedback chain is found to be linear by minimizing the quadratic quality criterion under additional Gaussian perturbations. Many results have been obtained showing the application of this approach to a wide range of engineering problems independent of the significant features of the engineering design of control system. At the initial stage of design a linear model of the controlled object is frequently used, especially in problems modelled by the equations of larger dimensions, and the quadratic quality criterion is sufficiently flexible. The desired characteristics of the system can be obtained by manipulating the parameters emerging in the criterion (see Bryson [1], Opdenacker *et al.* [1]). It should be noted that two approaches exist within the framework of the LQG-problem; the state space or the time domain method (see Sorenson [1]) and the method of transfer functions or frequency domain (see Mac Farlane [1]).

The existence of these two approaches (which are described in detail for example by Kailath [2] and by Patel and Munro [1]) is demonstrated in a problem of control of a multivariable stationary linear system where various methods are used in the description of controlled plant. Thus it is the representation by state-space equations, when the connection between input and output is described by a system of first order differential equations.

$$\dot{x} = Fx + Mu \qquad (0.1)$$
$$y = Lx$$

Here x – state vector, u-vector of control actions (input), y – vector of observable coordinates (output), F, M, L – constant matrices.

On the other hand, the representation of connection between input and output can be made in a form of systems of differential equations of the higher order

$$D_L\left(\frac{d}{dt}\right)y = N_L\left(\frac{d}{dt}\right)u \tag{0.2}$$

or

$$D_R\left(\frac{d}{dt}\right)\eta = u, \tag{0.3}$$

$$y = N_R\left(\frac{d}{dt}\right)\eta$$

η – vector of intermediate variable, $D_L(\cdot)$, $D_R(\cdot)$ – nonsingular (polynomial) matrices, $N_L(\cdot)$, $N_R(\cdot)$ are polynomial matrices of appropriate dimensions. In the terms of transfer functions (formal replacement d/dt on s), connection between an input and output looks so, accordingly

$$y = L(Es - F)^{-1}Mu, \tag{0.4}$$

$$y = D_L^{-1}(s)N_L(s)u,$$

$$y = N_R(s)D_R^{-1}(s)u$$

Here and further E is the identity matrix of appropriate dimensions. Both of the approaches are presented in a series of works (see Kailath [2], Patel and Munro [1], etc.). The difference between the approaches (in the first approach the object is described by a system of differential equations of the first order (0.1), and in the second the system of higher order equations (0.2), (0.3) is used) involves the application of different calculation procedures (for example, to solve algebraic Riccati equation (ARE), or to factorize rational matrices). Also it is essential that the frequency technique is associated with the parametrization of the set of the regulators which stabilize the object (see Larin [20]). These approaches replaced each other over time. Thus, since the 60^{ies} the time domain method has been developed intensively and reduces the problem of optimal regulator or filter construction to the solution of ARE (see Athans [1], Bryson and Ho-Yu-Chi [1], Doyle *et al.* [1], Kwakernaak and Sivan [1], and Sage and White [1]). Although there are obvious advantages of this approaches, there is a series of classes of stationary problems and applications, where the immediate application of standard procedure of the time domain method is certainly difficult (degenerate or coloured noise in the measurement channel, problems, where the seperation theorem (see Bryson, Ho-Yu-Chi [1]) or separation property (see Kailath [2]) can not be used, problems with delay in controllers etc.). This situation has been examined by Youla *et al.* [1] and the advantages of the frequency domain approach have been noted (based on Wiener-Hopf equation) in regulator synthesis. The following example was cited. Let the object motion be described by the stochastic equation

$$\dot{x} = Fx + Mu + \xi, \tag{0.5}$$

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \qquad M = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

It is necessary to find the regulator equation (transfer function K(s))

$$u = K(s)y, \qquad y = Lx + \Theta \tag{0.6}$$

$$L = [-1 \quad 1],$$

minimizing the quadratic quality criterion

$$I = \lim_{T \to \infty} \frac{1}{2T} \left\langle \int_{-T}^{T} (x'Rx + cu^2)\,\mathrm{d}t \right\rangle, \tag{0.7}$$

$$R = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \text{ and } c \text{ is scalar.}$$

The other values are as follows: x, u, ξ and Θ are phase vector, control actions, outer perturbation and noise in the measurement channel respectively. It is assumed that ξ and Θ are stationary Gaussian processes with covariational matrices

$$\langle \xi(t)\xi'(\tau) \rangle = \Psi\delta(t - \tau), \qquad \Psi = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\langle \Theta(t)\Theta(\tau) \rangle = \sigma_m^2\delta(t - \tau).$$

Here $\delta(t)$ is Dirak δ-function, $\langle\ \rangle$ is the calculation of the mathematical expectation. The prime (′) indicates transposition.

If $c > 0$, decision of a problem is divided on the decision of a problem of state feedback (briefly stated in p. 2 of Chapter 3) and problem of finding the estimate state (which is the output of the Kalman–Bucy filter). The above theorem, establishes this property, (the separation theorem), states: the decision of a problem of synthesis of optimum feedback law has a form

$$u = -c^{-1}M'S_1\hat{x}$$

Where the symmetric matrix S_1, being the solution of ARE

$$S_1F + F'S_1 - S_1Mc^{-1}M'S_1 + R = 0 \tag{0.8}$$

and $\hat{x}$ – the estimated state, determined by the following system of equations (Kalman–Bucy filter)

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = F\hat{x} + Mu + N(y - L\hat{x}), \quad N = \sigma_m^{-2}S_2L'$$

The matrix S_2 appearing in this relation, satisfies following ARE

$$FS_2 + S_2F - S_2L'\sigma_m^{-2}LS_2 + \Psi = 0 \tag{0.9}$$

In other words, here the transfer function of a controller has a form

$$K(s) = -c^{-1}M'S_1(Es - F + Mc^{-1}M'S_1 + S_2L'\sigma_m^{-2}L)^{-1}S_2L'\sigma_m^{-2}$$

If in (0.7) $c = 0$ the separation theorem does not "work", and loses the sense of the equation (0.8), but solution of the problem exists and has the form

$$K(s) = \frac{\left(\frac{7\sigma}{\sigma_m} + 12\right)s - \frac{2\sigma}{\sigma_m}}{s - \left(\frac{6\sigma}{\sigma_m} + 7\right)} \tag{0.10}$$

Yet if $\sigma_m = 0$, the equations (0.8) and (0.9) lose sense. But in this case a solution also exists and has the form:

$$K(s) = \frac{1}{3} - \frac{7}{6}s. \tag{0.11}$$

In Youla *et al.* [1] it is stressed that very simple regulators with transfer functions (0.10) and (0.11) can not be obtained by the standard procedures of the state space method. It is important that in Youla *et al.* [1] a key correlation of the modern frequency domain methods of synthesis is presented. Cheng and Pearson [1] are of the opinion that this is the most useful result of the frequency method of synthesis: the so-called parametrization of the set of regulator ensuring stability of the closed system "object + regulator". Let as in (0.4) the matrix of transfer function of the object $G(s)$ be represented as the "ratio" of two polynomial matrices (matrix-fraction description (MFD) Kailath [2], and Patel and Munro [1])

$$G(s) = D_L^{-1}(s)N_L(s) = N_R(s)D_R^{T^{-1}}(s) \tag{0.12}$$

The polynomial matrices $X(s)$ and $Y(s)$ satisfy the Diophantine equation

$$D_L(s)X(s) + N_L(s)Y(s) = E, \tag{0.13}$$

Then the set of all regulators $K(s)$ stabilizing the object (0.12) is described (parametrized) by the following relation (see Youla *et al.* [1])

$$K(s) = ((Y(s) + D_R(s)\Phi(s))(X(s) - N_R(s)\Phi(s))^{-1} \tag{0.14}$$

where the matrix $\Phi(s)$ is analytical for $\mathrm{Re}(s) \geq 0$. Relation (0.14) allows us to put down the optimizing functional (the expression (0.7)) in the frequency domain, i.e. to interpret it as a square of a norm in Hardi space H_2 of the corresponding transfer function, and therefore, to reduce the optimization problem to the choice of function $\Phi(s)$ which minimizes this norm. We recall that the Hardi space H_2 (see Glover [1], Grimble [2], Koosis [1] and Kucera [2]) consists of complex-valued function $f(s)$ being analytical on the open right half-plane and satisfying the condition

$$\|f\| = \sup_{\xi>0}\left[(2\pi)^{-1}\int_{\infty}^{\infty}|f(\xi + j\omega)|^2 d\omega\right]^{1/2} < \infty.$$

The Hardi space H_2 of matrices consists of the matrices $F(s)$ with elements from H_2 and the norm

$$\|F\|^2 = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} \operatorname{trace} F'(-s)F(s)\, \mathrm{d}s.$$

Similarly we determine the Hardi space H_2 over the unit circle. In scalar case this space of all series

$$f(z) = \sum_{k=0}^{\infty} f_k z^k, \qquad \sum_{k=0}^{\infty} |f_k|^2 < \infty$$

with the norm

$$\|f\|^2 = \sum_{k=0}^{\infty} |f_k|^2 = \frac{1}{2\pi i} \oint_{|z|=1} \left(f\left(\frac{1}{z}\right) f(z)\right) \frac{\mathrm{d}z}{z}$$

In matrix case the norm determined as follows

$$\|F(z)\|^2 = \frac{1}{2\pi i} \oint_{|z|=1} (\operatorname{trace} F(z)F'(z^{-1}))\, \frac{\mathrm{d}z}{z}$$

Note that there is an increasing interest in the frequency methods of synthesis. The idea seems appropriate, as presented in Kailath [2] that the descriptions of the object in terms of transfer functions or in terms of state space are in fact two extremes of the whole spectrum of the possible ways of describing finite dimensional system. Either description maybe used, though the situations are possible, when joint consideration of these forms seems natural. This idea (the combination of time and frequency approaches) is widely used in the present monograph in the development of both optimization procedures and numerical algorithms. The optimization procedures are based on the parametrization by (Larin *et al.* [1]) which is similar, as noted in (Park and Bongiorno [1]), to the parametrization (0.13) and (0.14), but are more versatile. The basic idea of it is as follows. The set of regulators which stabilize the object (0.12) is parametrized by (1.3–12) from (Larin *et al.* [1])

$$K(s) = -(B(s) + \Phi(s)N_L)^{-1}(\Phi(s)D_L(s) - A(s)), \tag{0.15}$$

where the polynomial matrices $A(s)$ and $B(s)$ ensure the Hurwits state or equality to the constant of determinant of the matrix

$$Z = \begin{bmatrix} D_L(s) & -N_L(s) \\ A(s) & B(s) \end{bmatrix} \tag{0.16}$$

Note that in (Larin *et al.* [1]) the relation (0.15) was obtained in the process of algorithmization of the procedure of choice of the variable function during the reducing the problem of optimization to the Wiener-Hopf equation. It is important that this approach proves to be efficient both as applied not only to the feedback systems and also in optimization problem, where the feedback chain is absent and thus, the notion of stability or stabilization of closed system is meaningless (see p. 3 sec. 1 of Chapter 2)

Naturally, in the case of feedback systems this approach enables the known algorithms of the stabilizing regulator set parametization (see Desoer *et al.* [1] and Youla *et al.* [1]) to be obtained and the series of new algorithms to be derived (for example, the matrices $A(s)$ and $B(s)$ may be found by the solution of ARE, rather than the Diophantine equation. When the matrices $A(s)$ and $B(s)$ satisfy the Diophantine equation

$$B(s)D_R(s) + A(s)N_R(s) = -E \tag{0.17}$$

then, as it is shown (see p. 2, Section 1 of Chapter 2) that parametrizations (0.15), (0.17) and (0.13), (0.14) are equivalent (see also Kailath [2], p. 540).

Before reviewing the contents of the monograph we note that the results cover the investigations (see Aliev, Bordyug *et al.* [15], Aliev, Larin *et al.* [1], Larin *et al.* [1, 3]) with the exception of control problems with delay (see Aliev, Bordyug *et al.* [1], Aliev, Larin *et al.* [1]) which have continue in (Aliev, Bordyug *et al.* [16], Aliev, Larin [2, 4–7] and Larin, Aliev [2]). Thus, Chapters 1 and 2 present analytical methods and the second part of the monograph (Chapter 3, 4) deals with the development of numerical algorithms (the list of which is given in the contents list) covering the computing procedures of the state space method and frequency domain technique. Note that the original algorithms are presented and as a rule, these do not duplicate those cited in (Laub [2] and Petkov *et al.* [2]).

In Chapter 1, the state space method procedures are modified to solve the LQG-problem in non-standard conditions. The first section of this Chapter is introductory and presents known results of LQG-problem for continuous and discrete time including the case of steady state.

Further, the solution of the LQG problem is constructed using an asymptotic approach in the case of continuous time and steady state, when the restriction on the controllers and the noise intensity in the measurement channel tend to be zero (as applied to the above-cited example see Youla *et al.* [1]) this corresponds to the cases $c \to 0$ and $\sigma_m \to 0$). In this case, the solution of the problem is reduced to the construction of solution to ARE of smaller dimensions. However, the traditional approach does not work properly in every case, for example, when the eigenvalues of Hamiltonian matrix are close to the imaginary axis (quasiconservative or weakly damped systems). In this regard a special asymptotic procedure is described for the construction and refinement of an approximate solution to the problem of weak control of weakly damped systems. It is known that in such problems, the approximate solution (after substitution into ARE) does not minimize the norm of residual matrix and therefore special techniques are incorporated to construct asymptotic expansion to the solution.

Further, in the fourth section of chapter 1 a special case of the linear quadratic problem is described where the Hamiltonian matrix possesses zero eigenvalues. This complicates the construction of the desired solution to ARE in such problems. The last section deals with the problem of optimal regulator syntheses, when the object motion is described by a periodic system of differential and difference equations or difference ones. Such problems are associated with the problem of design of vehicles on magnetic cushion, biped apparatus. The algorithm presented here allows the solution to the problem in the case of a singular matrix as well that which specifies the quadratic form of the control actions in the optimized functional. Naturally, this algorithm is also efficient when the above matrix is ill-conditioned.

Chapter 2 is devoted to the development of the frequency domain method of synthesis. Here the frequency method is presented based on Wiener-Hopf equation and minimization of H_2-norm.

The first section of Chapter 2 details the parametrization of the form of (0.15), (0.16). This procedure is compared with that of (0.12), (0.14). It is noted that the parametrization problem emerges as well in the solution of the problems independent of the synthesis of optimal feedback chain.

In the second section, the solution of the problem of optimal regulator synthesis is introduced and algorithms of the polynomial approach are shown to follow from the presented solution under certain restrictions of the class of problems under consideration (the coloured noise can not be presented in the measurement channel). Also, a generalized version of the factorizational Anderson-Moor identity is presented.

The expansion of applications of the frequency domain method is due to the formulation of problems in terms of minimization of various norms (H_∞, H_2, l^1 etc.) Moreover, a certain unification of the statements is now apparent (for example of the standard problem (see Doyle *et al.* [1]), model matching problem (see Sebryakov and Semenov [2], etc.). According to this technique, by using the parametrization (see Desoer *et al.* [1] and Youla *et al.* [1]) of the regulator set ensuring stability of the closed system, the inital problem is reduced to a model matching problem the solution of which is obtained by minimization of a norm.

The third section of Chapter 2, the problem of stabilization is formalized in terms of a standard problem based on parametrization (0.15), (0.16). Regulator (0.15) is also proved to stabilize the object when in (0.16) the matrices $D_L(s)$ and $N_L(s)$ are not polynomial (generalized version of parametrization (see Desoer *et al.* [1])). Further, the problem is formulated, in terms of a model matching problem and its solution is found to correspond to the H_2-norm minimization. The independence of the solution (the transfer function of optimal regulator) of the **concrete** choice of the matrices $A(s)$ and $B(s)$ in (0.16) is shown. In conclusion, the optimization problem under non-zero initial conditions is treated as applied to the singular systems. In last section, the so-called systems with several degrees of freedom are analysed. The problem of the servomechanism synthesis has two degrees of freedom by definition, and three degrees of freedom when the outer perturbation is measured, etc. It is essential that in each case an appropriate parametrization of the regulator set is constructed. Also based on the approaches (Aliev, Larin *et al.* [1], and Larin *et al.* [3]) it is shown how the corresponding parametrization may be obtained from the parametrization (0.15), (0.16).