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PURE AND APPLIED MATHEMATICS SERIES

RANDOM MATRICES

THIRD EDITION

MADAN LAL MEHTA

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Third Edition

Madan Lal Mehta

Saclay, Gif-sur-Yvette, France

ELSEVIER
ACADEMIC
PRESS

Amsterdam Boston Heidelberg London New York Oxford
Paris San Diego San Francisco Singapore Sydney Tokyo

ELSEVIER B.V.
Sara Burgerhartstraat 25
P.O. Box 211, 1000 AE Amsterdam,
The Netherlands

ELSEVIER Inc.
525 B Street, Suite 1900
San Diego, CA 92101-4495, USA

ELSEVIER Ltd
The Boulevard, Langford Lane
Kidlington, Oxford OX5 1GB, UK

ELSEVIER Ltd
84 Theobalds Road
London WC1X 8RR UK

First edition 2004

Library of Congress Cataloging in Publication Data
A catalog record is available from the Library of Congress.

British Library Cataloguing in Publication Data
A catalogue record is available from the British Library.

Series ISSN 0079-8169
ISBN: 0-12-088409-7

⊗ The paper used in this publication meets the requirements of ANSI/NISO Z39.48-1992 (Permanence of Paper).
Printed in the Netherlands.

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This is volume 142 in the PURE AND APPLIED MATHEMATICS series
Founding Editors: Paul A. Smith and Samuel Eilenberg

PREFACE TO THE THIRD EDITION

In the last decade following the publication of the second edition of this book the subject of random matrices found applications in many new fields of knowledge. In heterogeneous conductors (mesoscopic systems) where the passage of electric current may be studied by transfer matrices, quantum chromo dynamics characterized by some Dirac operator, quantum gravity modeled by some random triangulation of surfaces, traffic and communication networks, zeta function and L-series in number theory, even stock movements in financial markets, wherever imprecise matrices occurred, people dreamed of random matrices.

Some new analytical results were also added to the random matrix theory. The noteworthy of them being, the awareness that certain Fredholm determinants satisfy second order nonlinear differential equations, power series expansion of spacing functions, a compact expression (one single determinant) of the general correlation function for the case of hermitian matrices coupled in a chain, probability densities of random determinants, and relation to random permutations. Consequently, a revision of this book was felt necessary, though in the mean time four new books (Girko, 1990; Effetof, 1997; Katz and Sarnak, 1999; Deift, 2000), two long review articles (di Francesco et al., 1995; Guhr et al., 1998) and a special issue of J. Phys. A (2003) have appeared. The subject matter of them is either complimentary or disjoint. Apart from them the introductory article by C.E. Porter in his 1965 collection of reprints remains instructive even today.

In this new edition most chapters remain almost unaltered though some of them change places. Chapter 5 is new explaining the basic tricks of the trade, how to deal with integrals containing the product of differences $\prod |x_i - x_j|$ raised to the power 1, 2 or 4. Old Chapters 5 to 11 shift by one place to become Chapters 6 to 12, while Chapter 12 becomes 18. In Chapter 15 two new sections dealing with real random matrices and the probability density of determinants are added. Chapters 20 to 27 are new. Among the appendices some have changed places or were regrouped, while 16, 37, 38 and 42 to 54 are new. One major and some minor errors have been corrected. It is really surprising how such a major error could have creeped in and escaped detection by so many experts reading it. (Cf. lines 2, 3 after Eq. (1.8.15) and line 6 after Eq. (1.8.16); $h(d)$ is not the number of different quadratic forms as presented, but is the number of different primitive inequivalent quadratic forms.) Not to hinder the fluidity of reading the original source of the material presented is rarely indicated in the text. This is done in the "notes" at the end.

While preparing this new edition I remembered the comment of B. Suderland that from the presentation point of view he preferred the first edition rather than the second. As usual, I had free access to the published and unpublished works of my teachers, colleagues and friends F.J. Dyson, M. Gaudin, H. Widom, C.A. Tracy, A.M. Odlyzko, B. Poonen, H.S. Wilf, A. Edelman, B. Dietz, S. Ghosh, B. Eynard, R.A. Askey and many others. G. Mahoux kindly wrote Appendix A.16. M. Gingold helped me in locating some references and L. Bervas taught me how to use a computer to incorporate a figure as a .ps file in the TEX files of the text. G. Cicuta, O. Bohigas, B. Dietz, M. Gaudin, S. Ghosh, P.B. Kahn, G. Mahoux, J.-M. Normand, N.C. Snaith, P. Sarnak, H. Widom and R. Conte read portions of the manuscript and made critical comments thus helping me to avoid errors, inaccuracies and even some blunders. O. Bohigas kindly supplied me with a list of minor errors of references in the figures of Chapter 16. It is my pleasant duty to thank all of them. However, the responsibility of any remaining errors is entirely mine. Hopefully this new edition is free of serious errors and it is self-contained to be accessible to any diligent reader.

February, 2004
Saclay, France

Madan Lal MEHTA

PREFACE TO THE SECOND EDITION

The contemporary textbooks on classical or quantum mechanics deal with systems governed by differential equations which are simple enough to be solved in closed terms (or eventually perturbatively). Hence the entire past and future of such systems can be deduced from a knowledge of their present state (initial conditions). Moreover, these solutions are stable in the sense that small changes in the initial conditions result in small changes in their time evolution. Such systems are called integrable. Physicists and mathematicians now realize that most of the systems in nature are not integrable. The forces and interactions are so complicated that either we can not write the corresponding differential equation, or when we can, the whole situation is unstable; a small change in the initial conditions produces a large difference in the final outcome. They are called chaotic. The relation of chaotic to integrable systems is something like that of transcendental to rational numbers.

For chaotic systems it is meaningless to calculate the future evolution starting from an exactly given present state, because a small error or change at the beginning will make the whole computation useless. One should rather try to determine the statistical properties of such systems.

The theory of random matrices makes the hypothesis that the characteristic energies of chaotic systems behave locally as if they were the eigenvalues of a matrix with randomly distributed elements. Random matrices were first encountered in mathematical statistics by Hsu, Wishart and others in the 1930s, but an intensive study of their properties in connection with nuclear physics began only with the work of Wigner in the 1950s. In 1965 C.E. Porter edited a reprint volume of all important papers on the subject, with a critical and detailed introduction which even today is very instructive. The first edition of the present book appeared in 1967. During the last two decades many new results have been discovered, and a larger number of physicists and mathematicians got interested in the subject owing to various possible applications. Consequently it was felt that this book has to be revised even though a nice review article by Brody et al. has appeared in the mean time (*Rev. Mod. Phys.*, 1981).

Among the important new results one notes the theory of matrices with quaternion elements which serves to compute some multiple integrals, the evaluation of n-point spacing probabilities, the derivation of the asymptotic behaviour of nearest neighbor spacings, the computation of a few hundred millions of zeros of the Riemann zeta function and the analysis of their statistical properties, the rediscovery of Selberg's 1944 paper giving rise to hundreds of recent publications, the use of the diffusion equation to evaluate an integral over the unitary group thus allowing the analysis of non-invariant Gaussian ensembles and the numerical investigation of various systems with deterministic chaos.

After a brief survey of the symmetry requirements the Gaussian ensembles of random Hermitian matrices are introduced in Chapter 2. In Chapter 3 the joint probability density of the eigenvalues of such matrices is derived. In Chapter 5 we give a detailed treatment of the simplest of the matrix ensembles, the Gaussian unitary one, deriving the n-point correlation functions and the n-point spacing probabilities. Here we explain how the Fredholm theory of integral equations can be used to derive the limits of large determinants. In Chapter 6 we study the Gaussian orthogonal ensemble which in most cases is appropriate for applications but is mathematically more complicated. Here we introduce matrices with quaternion elements and their determinants as well as the method of integration over alternate variables. The short Chapter 8 introduces a Brownian motion model of Gaussian Hermitian matrices. Chapters 9, 10 and 11 deal with ensembles of unitary random matrices, the mathematical methods being the same as in Chapters 5 and 6. In Chapter 12 we derive the asymptotic series for the nearest neighbor spacing probability. In Chapter 14 we study a non-invariant Gaussian Hermitian ensemble, deriving its n-point correlation and cluster functions; it is a good example of the use of mathematical tools developed in Chapters 5 and 6. Chapter 16 describes a number of statistical quantities useful for the analysis of experimental data. Chapter 17 gives a detailed account of Selberg's integral and of its consequences. Other chapters deal with questions or ensembles less important either for applications or for the mathematical methods used. Numerous appendices treat secondary mathematical questions, list power series expansions and numerical tables of various functions useful in applications.

The methods explained in Chapters 5 and 6 are basic, they are necessary to understand most of the material presented here. However, Chapter 17 is independent. Chapter 12 is the most difficult one, since it uses results from the asymptotic analysis of differential equations, Toeplitz determinants and the inverse scattering theory, for which in spite of a few nice references we are unaware of a royal road. The rest of the material is self-contained and hopefully quite accessible to any diligent reader with modest mathematical background.

Contrary to the general tendency these days, this book contains no exercises.

October, 1990 M.L. MEHTA
Saclay, France

PREFACE TO THE FIRST EDITION

Though random matrices were first encountered in mathematical statistics by Hsu, Wishart, and others, intensive study of their properties in connection with nuclear physics began with the work of Wigner in the 1950s. Much material has accumulated since then, and it was felt that it should be collected. A reprint volume to satisfy this need had been edited by C.E. Porter with a critical introduction (see References); nevertheless, the feeling was that a book containing a coherent treatment of the subject would be welcome.

We make the assumption that the local statistical behavior of the energy levels of a sufficiently complicated system is simulated by that of the eigenvalues of a random matrix. Chapter 1 is a rapid survey of our understanding of nuclear spectra from this point of view. The discussion is rather general, in sharp contrast to the precise problems treated in the rest of the book. In Chapter 2 an analysis of the usual symmetries that quantum system might possess is carried out, and the joint probability density function for the various matrix elements of the Hamiltonian is derived as a consequence. The transition from matrix elements to eigenvalues is made in Chapter 3, and the standard arguments of classical statistical mechanics are applied in Chapter 4 to derive the eigenvalue density. An unproven conjecture is also stated. In Chapter 5 the method of integration over alternate variables is presented, and an application of the Fredholm theory of integral equations is made to the problem of eigenvalue spacings. The methods developed in Chapter 5 are basic to an understanding of most of the remaining chapters. Chapter 6 deals with the correlations and spacings for less useful cases. A Brownian motion model is described in Chapter 7. Chapters 8 to 11 treat circular ensembles; Chapters 8 to 10 repeat calculations analogous to those of Chapter 4 to 7. The integration method discussed in Chapter 11 originated with Wigner and is being published here for the first time. The theory of non-Hermitian random matrices, though not applicable to any physical problems, is a fascinating subject and must be studied for its own sake. In this direction an impressive effort by Ginibre is described in Chapter 12. For the Gaussian ensembles the level density in regions where it is very low is discussed in Chapter 13. The investigations of Chapter 16 and Appendices A.29 and A.30 were recently carried out in collaboration with Professor Wigner at Princeton University. Chapters 14, 15, and 17 treat a number of other topics. Most of the material in the appendices is either well known or was published elsewhere and is collected here for ready reference. It was surprisingly difficult to obtain the proof contained in Appendix A.21, while Appendices A.29, A.30 and A.31 are new.

October, 1967 M.L. MEHTA
Saclay, France

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