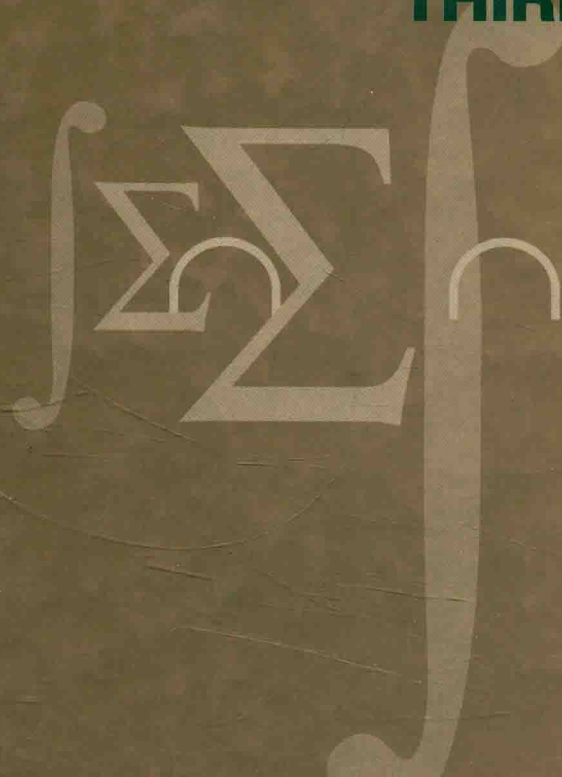


PURE AND APPLIED MATHEMATICS SERIES



RANDOM MATRICES

THIRD EDITION



MADAN LAL MEHTA

RANDOM MATRICES

Third Edition

Madan Lal Mehta

Saclay, Gif-sur-Yvette, France



Amsterdam Boston Heidelberg London New York Oxford
Paris San Diego San Francisco Singapore Sydney Tokyo

ELSEVIER B.V.

Sara Burgerhartstraat 25
P.O. Box 211, 1000 AE Amsterdam,
The Netherlands

ELSEVIER Ltd
The Boulevard, Langford Lane
Kidlington, Oxford OX5 1GB, UK

ELSEVIER Inc.

525 B Street, Suite 1900
San Diego, CA 92101-4495, USA

ELSEVIER Ltd
84 Theobalds Road
London WC1X 8RR UK

© 2004 Elsevier Ltd. All rights reserved.

This work is protected under copyright by Elsevier, and the following terms and conditions apply to its use:

Photocopying

Single photocopies of single chapters may be made for personal use as allowed by national copyright laws. Permission of the Publisher and payment of a fee is required for all other photocopying, including multiple or systematic copying, copying for advertising or promotional purposes, resale, and all forms of document delivery. Special rates are available for educational institutions that wish to make photocopies for non-profit educational classroom use.

Permissions may be sought directly from Elsevier's Science & Technology Rights Department in Oxford, UK: phone: (+44) 1865 843830, fax: (+44) 1865 853333, e-mail: permissions@elsevier.com. You may also complete your request on-line via the Elsevier homepage (<http://www.elsevier.com>), by selecting 'Customer Support' and then 'Obtaining Permissions'.

In the USA, users may clear permissions and make payments through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA; phone: (+1) (978) 7508400, fax: (+1) (978) 7504744, and in the UK through the Copyright Licensing Agency Rapid Clearance Service (CLARCS), 90 Tottenham Court Road, London W1P 0LP, UK; phone: (+44) 207 631 5555; fax: (+44) 207 631 5500. Other countries may have a local reprographic rights agency for payments.

Derivative Works

Tables of contents may be reproduced for internal circulation, but permission of Elsevier is required for external resale or distribution of such material.

Permission of the Publisher is required for all other derivative works, including compilations and translations.

Electronic Storage or Usage

Permission of the Publisher is required to store or use electronically any material contained in this work, including any chapter or part of a chapter.

Except as outlined above, no part of this work may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without prior written permission of the Publisher. Address permissions requests to: Elsevier's Science & Technology Rights Department, at the phone, fax and e-mail addresses noted above.

Notice

No responsibility is assumed by the Publisher for any injury and/or damage to persons or property as a matter of products liability, negligence or otherwise, or from any use or operation of any methods, products, instructions or ideas contained in the material herein. Because of rapid advances in the medical sciences, in particular, independent verification of diagnoses and drug dosages should be made.

First edition 2004

Library of Congress Cataloging in Publication Data

A catalog record is available from the Library of Congress.

British Library Cataloguing in Publication Data

A catalogue record is available from the British Library.

Series ISSN 0079-8169

ISBN: 0-12-088409-7

© The paper used in this publication meets the requirements of ANSI/NISO Z39.48-1992 (Permanence of Paper).

Printed in the Netherlands.

Working together to grow
libraries in developing countries

www.elsevier.com | www.bookaid.org | www.sabre.org

ELSEVIER

BOOK AID
International

Sabre Foundation

RANDOM MATRICES

Third Edition

This is volume 142 in the PURE AND APPLIED MATHEMATICS series
Founding Editors: Paul A. Smith and Samuel Eilenberg

PREFACE TO THE THIRD EDITION

In the last decade following the publication of the second edition of this book the subject of random matrices found applications in many new fields of knowledge. In heterogeneous conductors (mesoscopic systems) where the passage of electric current may be studied by transfer matrices, quantum chromo dynamics characterized by some Dirac operator, quantum gravity modeled by some random triangulation of surfaces, traffic and communication networks, zeta function and L -series in number theory, even stock movements in financial markets, wherever imprecise matrices occurred, people dreamed of random matrices.

Some new analytical results were also added to the random matrix theory. The noteworthy of them being, the awareness that certain Fredholm determinants satisfy second order nonlinear differential equations, power series expansion of spacing functions, a compact expression (one single determinant) of the general correlation function for the case of hermitian matrices coupled in a chain, probability densities of random determinants, and relation to random permutations. Consequently, a revision of this book was felt necessary, though in the mean time four new books (Girko, 1990; Effetof, 1997; Katz and Sarnak, 1999; Deift, 2000), two long review articles (di Francesco et al., 1995; Guhr et al., 1998) and a special issue of J. Phys. A (2003) have appeared. The subject matter of them is either complimentary or disjoint. Apart from them the introductory article by C.E. Porter in his 1965 collection of reprints remains instructive even today.

In this new edition most chapters remain almost unaltered though some of them change places. Chapter 5 is new explaining the basic tricks of the trade, how to deal with integrals containing the product of differences $\prod |x_i - x_j|$ raised to the power 1, 2 or 4. Old Chapters 5 to 11 shift by one place to become Chapters 6 to 12, while Chapter 12 becomes 18. In Chapter 15 two new sections dealing with real random matrices and the probability density of determinants are added. Chapters 20 to 27 are new. Among the appendices some have changed places or were regrouped, while 16, 37, 38

and 42 to 54 are new. One major and some minor errors have been corrected. It is really surprising how such a major error could have crept in and escaped detection by so many experts reading it. (Cf. lines 2, 3 after Eq. (1.8.15) and line 6 after Eq. (1.8.16); $h(d)$ is not the number of different quadratic forms as presented, but is the number of different primitive inequivalent quadratic forms.) Not to hinder the fluidity of reading the original source of the material presented is rarely indicated in the text. This is done in the “notes” at the end.

While preparing this new edition I remembered the comment of B. Suderland that from the presentation point of view he preferred the first edition rather than the second. As usual, I had free access to the published and unpublished works of my teachers, colleagues and friends F.J. Dyson, M. Gaudin, H. Widom, C.A. Tracy, A.M. Odlyzko, B. Poonen, H.S. Wilf, A. Edelman, B. Dietz, S. Ghosh, B. Eynard, R.A. Askey and many others. G. Mahoux kindly wrote Appendix A.16. M. Gingold helped me in locating some references and L. Bervas taught me how to use a computer to incorporate a figure as a .ps file in the \TeX files of the text. G. Cicuta, O. Bohigas, B. Dietz, M. Gaudin, S. Ghosh, P.B. Kahn, G. Mahoux, J.-M. Normand, N.C. Snaith, P. Sarnak, H. Widom and R. Conte read portions of the manuscript and made critical comments thus helping me to avoid errors, inaccuracies and even some blunders. O. Bohigas kindly supplied me with a list of minor errors of references in the figures of Chapter 16. It is my pleasant duty to thank all of them. However, the responsibility of any remaining errors is entirely mine. Hopefully this new edition is free of serious errors and it is self-contained to be accessible to any diligent reader.

February, 2004
Saclay, France

Madan Lal MEHTA

PREFACE TO THE SECOND EDITION

The contemporary textbooks on classical or quantum mechanics deal with systems governed by differential equations which are simple enough to be solved in closed terms (or eventually perturbatively). Hence the entire past and future of such systems can be deduced from a knowledge of their present state (initial conditions). Moreover, these solutions are stable in the sense that small changes in the initial conditions result in small changes in their time evolution. Such systems are called integrable. Physicists and mathematicians now realize that most of the systems in nature are not integrable. The forces and interactions are so complicated that either we can not write the corresponding differential equation, or when we can, the whole situation is unstable; a small change in the initial conditions produces a large difference in the final outcome. They are called chaotic. The relation of chaotic to integrable systems is something like that of transcendental to rational numbers.

For chaotic systems it is meaningless to calculate the future evolution starting from an exactly given present state, because a small error or change at the beginning will make the whole computation useless. One should rather try to determine the statistical properties of such systems.

The theory of random matrices makes the hypothesis that the characteristic energies of chaotic systems behave locally as if they were the eigenvalues of a matrix with randomly distributed elements. Random matrices were first encountered in mathematical statistics by Hsu, Wishart and others in the 1930s, but an intensive study of their properties in connection with nuclear physics began only with the work of Wigner in the 1950s. In 1965 C.E. Porter edited a reprint volume of all important papers on the subject, with a critical and detailed introduction which even today is very instructive. The first edition of the present book appeared in 1967. During the last two decades many new results have been discovered, and a larger number of physicists and mathematicians got interested in the subject owing to various possible applications. Consequently it was felt that this book has to be revised even though a nice review article by Brody et al. has appeared in the mean time (*Rev. Mod. Phys.*, 1981).

Among the important new results one notes the theory of matrices with quaternion elements which serves to compute some multiple integrals, the evaluation of n -point spacing probabilities, the derivation of the asymptotic behaviour of nearest neighbor spacings, the computation of a few hundred millions of zeros of the Riemann zeta function and the analysis of their statistical properties, the rediscovery of Selberg's 1944 paper giving rise to hundreds of recent publications, the use of the diffusion equation to evaluate an integral over the unitary group thus allowing the analysis of non-invariant Gaussian ensembles and the numerical investigation of various systems with deterministic chaos.

After a brief survey of the symmetry requirements the Gaussian ensembles of random Hermitian matrices are introduced in Chapter 2. In Chapter 3 the joint probability density of the eigenvalues of such matrices is derived. In Chapter 5 we give a detailed treatment of the simplest of the matrix ensembles, the Gaussian unitary one, deriving the n -point correlation functions and the n -point spacing probabilities. Here we explain how the Fredholm theory of integral equations can be used to derive the limits of large determinants. In Chapter 6 we study the Gaussian orthogonal ensemble which in most cases is appropriate for applications but is mathematically more complicated. Here we introduce matrices with quaternion elements and their determinants as well as the method of integration over alternate variables. The short Chapter 8 introduces a Brownian motion model of Gaussian Hermitian matrices. Chapters 9, 10 and 11 deal with ensembles of unitary random matrices, the mathematical methods being the same as in Chapters 5 and 6. In Chapter 12 we derive the asymptotic series for the nearest neighbor spacing probability. In Chapter 14 we study a non-invariant Gaussian Hermitian ensemble, deriving its n -point correlation and cluster functions; it is a good example of the use of mathematical tools developed in Chapters 5 and 6. Chapter 16 describes a number of statistical quantities useful for the analysis of experimental data. Chapter 17 gives a detailed account of Selberg's integral and of its consequences. Other chapters deal with questions or ensembles less important either for applications or for the mathematical methods used. Numerous appendices treat secondary mathematical questions, list power series expansions and numerical tables of various functions useful in applications.

The methods explained in Chapters 5 and 6 are basic, they are necessary to understand most of the material presented here. However, Chapter 17 is independent. Chapter 12 is the most difficult one, since it uses results from the asymptotic analysis of differential equations, Toeplitz determinants and the inverse scattering theory, for which in spite of a few nice references we are unaware of a royal road. The rest of the material is self-contained and hopefully quite accessible to any diligent reader with modest mathematical background.

Contrary to the general tendency these days, this book contains no exercises.

October, 1990
Saclay, France

M.L. MEHTA

PREFACE TO THE FIRST EDITION

Though random matrices were first encountered in mathematical statistics by Hsu, Wishart, and others, intensive study of their properties in connection with nuclear physics began with the work of Wigner in the 1950s. Much material has accumulated since then, and it was felt that it should be collected. A reprint volume to satisfy this need had been edited by C.E. Porter with a critical introduction (see References); nevertheless, the feeling was that a book containing a coherent treatment of the subject would be welcome.

We make the assumption that the local statistical behavior of the energy levels of a sufficiently complicated system is simulated by that of the eigenvalues of a random matrix. Chapter 1 is a rapid survey of our understanding of nuclear spectra from this point of view. The discussion is rather general, in sharp contrast to the precise problems treated in the rest of the book. In Chapter 2 an analysis of the usual symmetries that quantum system might possess is carried out, and the joint probability density function for the various matrix elements of the Hamiltonian is derived as a consequence. The transition from matrix elements to eigenvalues is made in Chapter 3, and the standard arguments of classical statistical mechanics are applied in Chapter 4 to derive the eigenvalue density. An unproven conjecture is also stated. In Chapter 5 the method of integration over alternate variables is presented, and an application of the Fredholm theory of integral equations is made to the problem of eigenvalue spacings. The methods developed in Chapter 5 are basic to an understanding of most of the remaining chapters. Chapter 6 deals with the correlations and spacings for less useful cases. A Brownian motion model is described in Chapter 7. Chapters 8 to 11 treat circular ensembles; Chapters 8 to 10 repeat calculations analogous to those of Chapter 4 to 7. The integration method discussed in Chapter 11 originated with Wigner and is being published here for the first time. The theory of non-Hermitian random matrices, though not applicable to any physical problems, is a fascinating subject and must be studied for its

own sake. In this direction an impressive effort by Ginibre is described in Chapter 12. For the Gaussian ensembles the level density in regions where it is very low is discussed in Chapter 13. The investigations of Chapter 16 and Appendices A.29 and A.30 were recently carried out in collaboration with Professor Wigner at Princeton University. Chapters 14, 15, and 17 treat a number of other topics. Most of the material in the appendices is either well known or was published elsewhere and is collected here for ready reference. It was surprisingly difficult to obtain the proof contained in Appendix A.21, while Appendices A.29, A.30 and A.31 are new.

October, 1967
Saclay, France

M.L. MEHTA

CONTENTS

| | |
|---|--------|
| Preface to the Third Edition | xiii |
| Preface to the Second Edition | xv |
| Preface to the First Edition | xvii |
| Chapter 1. Introduction | 1 |
| 1.1. Random Matrices in Nuclear Physics | 1 |
| 1.2. Random Matrices in Other Branches of Knowledge | 5 |
| 1.3. A Summary of Statistical Facts about Nuclear Energy Levels | 8 |
| 1.3.1. Level Density | 8 |
| 1.3.2. Distribution of Neutron Widths | 9 |
| 1.3.3. Radiation and Fission Widths | 9 |
| 1.3.4. Level Spacings | 10 |
| 1.4. Definition of a Suitable Function for the Study of Level Correlations | 10 |
| 1.5. Wigner Surmise | 13 |
| 1.6. Electromagnetic Properties of Small Metallic Particles | 15 |
| 1.7. Analysis of Experimental Nuclear Levels | 16 |
| 1.8. The Zeros of The Riemann Zeta Function | 16 |
| 1.9. Things Worth Consideration, But Not Treated in This Book | 30 |
| Chapter 2. Gaussian Ensembles. The Joint Probability Density Function for the Matrix Elements | 33 |
| 2.1. Preliminaries | 33 |
| 2.2. Time-Reversal Invariance | 34 |
| 2.3. Gaussian Orthogonal Ensemble | 36 |
| 2.4. Gaussian Symplectic Ensemble | 38 |
| 2.5. Gaussian Unitary Ensemble | 42 |
| 2.6. Joint Probability Density Function for the Matrix Elements | 43 |
| 2.7. Gaussian Ensemble of Hermitian Matrices With Unequal Real and Imaginary Parts | 48 |
| 2.8. Anti-Symmetric Hermitian Matrices | 48 |
| Summary of Chapter 2 | 49 |

| | | |
|------------|--|-----|
| Chapter 3. | Gaussian Ensembles. The Joint Probability Density Function for the Eigenvalues . . . | 50 |
| 3.1. | Orthogonal Ensemble | 50 |
| 3.2. | Symplectic Ensemble | 54 |
| 3.3. | Unitary Ensemble | 56 |
| 3.4. | Ensemble of Anti-Symmetric Hermitian Matrices | 59 |
| 3.5. | Gaussian Ensemble of Hermitian Matrices With Unequal Real and Imaginary Parts | 60 |
| 3.6. | Random Matrices and Information Theory | 60 |
| | Summary of Chapter 3 | 62 |
| Chapter 4. | Gaussian Ensembles Level Density | 63 |
| 4.1. | The Partition Function | 63 |
| 4.2. | The Asymptotic Formula for the Level Density. Gaussian Ensembles | 65 |
| 4.3. | The Asymptotic Formula for the Level Density. Other Ensembles | 67 |
| | Summary of Chapter 4 | 69 |
| Chapter 5. | Orthogonal, Skew-Orthogonal and Bi-Orthogonal Polynomials | 71 |
| 5.1. | Quaternions, Pfaffians, Determinants | 72 |
| 5.2. | Average Value of $\prod_{j=1}^N f(x_j)$; Orthogonal and Skew-Orthogonal Polynomials | 77 |
| 5.3. | Case $\beta = 2$; Orthogonal Polynomials | 78 |
| 5.4. | Case $\beta = 4$; Skew-Orthogonal Polynomials of Quaternion Type | 82 |
| 5.5. | Case $\beta = 1$; Skew-Orthogonal Polynomials of Real Type | 84 |
| 5.6. | Average Value of $\prod_{j=1}^N \psi(x_j, y_j)$; Bi-Orthogonal Polynomials | 88 |
| 5.7. | Correlation Functions | 89 |
| 5.8. | Proof of Theorem 5.7.1 | 93 |
| 5.8.1. | Case $\beta = 2$ | 93 |
| 5.8.2. | Case $\beta = 4$ | 94 |
| 5.8.3. | Case $\beta = 1$, Even Number of Variables | 96 |
| 5.8.4. | Case $\beta = 1$, Odd Number of Variables | 99 |
| 5.9. | Spacing Functions | 101 |
| 5.10. | Determinantal Representations | 101 |
| 5.11. | Integral Representations | 103 |
| 5.12. | Properties of the Zeros | 106 |
| 5.13. | Orthogonal Polynomials and the Riemann–Hilbert Problem | 107 |
| 5.14. | A Remark (Balian) | 108 |
| | Summary of Chapter 5 | 108 |
| Chapter 6. | Gaussian Unitary Ensemble | 110 |
| 6.1. | Generalities | 111 |
| 6.1.1. | About Correlation and Cluster Functions. | 111 |
| 6.1.2. | About Level-Spacings. | 113 |
| 6.1.3. | Spacing Distribution. | 118 |
| 6.1.4. | Correlations and Spacings. | 118 |
| 6.2. | The n -Point Correlation Function | 118 |
| 6.3. | Level Spacings | 122 |
| 6.4. | Several Consecutive Spacings | 127 |
| 6.5. | Some Remarks | 134 |
| | Summary of Chapter 6 | 144 |

| | | |
|-------------|--|-----|
| Chapter 7. | Gaussian Orthogonal Ensemble | 146 |
| 7.1. | Generalities | 147 |
| 7.2. | Correlation and Cluster Functions | 148 |
| 7.3. | Level Spacings. Integration Over Alternate Variables | 154 |
| 7.4. | Several Consecutive Spacings: $n = 2r$ | 157 |
| 7.5. | Several Consecutive Spacings: $n = 2r - 1$ | 162 |
| 7.5.1. | Case $n = 1$ | 163 |
| 7.5.2. | Case $n = 2r - 1$ | 164 |
| 7.6. | Bounds for the Distribution Function of the Spacings | 168 |
| | Summary of Chapter 7 | 172 |
| Chapter 8. | Gaussian Symplectic Ensemble | 175 |
| 8.1. | A Quaternion Determinant | 175 |
| 8.2. | Correlation and Cluster Functions | 177 |
| 8.3. | Level Spacings | 179 |
| | Summary of Chapter 8 | 181 |
| Chapter 9. | Gaussian Ensembles: Brownian Motion Model | 182 |
| 9.1. | Stationary Ensembles | 182 |
| 9.2. | Nonstationary Ensembles | 183 |
| 9.3. | Some Ensemble Averages | 187 |
| | Summary of Chapter 9 | 189 |
| Chapter 10. | Circular Ensembles | 191 |
| 10.1. | Orthogonal Ensemble | 192 |
| 10.2. | Symplectic Ensemble | 194 |
| 10.3. | Unitary Ensemble | 196 |
| 10.4. | The Joint Probability Density of the Eigenvalues | 197 |
| | Summary of Chapter 10 | 201 |
| Chapter 11. | Circular Ensembles (Continued) | 203 |
| 11.1. | Unitary Ensemble. Correlation and Cluster Functions | 203 |
| 11.2. | Unitary Ensemble. Level Spacings | 205 |
| 11.3. | Orthogonal Ensemble. Correlation and Cluster Functions | 207 |
| 11.3.1. | The Case $N = 2m$, Even | 209 |
| 11.3.2. | The Case $N = 2m + 1$, Odd | 210 |
| 11.3.3. | Conditions of Theorem 5.1.4 | 211 |
| 11.3.4. | Correlation and Cluster Functions | 212 |
| 11.4. | Orthogonal Ensemble. Level Spacings | 213 |
| 11.5. | Symplectic Ensemble. Correlation and Cluster Functions | 216 |
| 11.6. | Relation Between Orthogonal and Symplectic Ensembles | 218 |
| 11.7. | Symplectic Ensemble. Level Spacings | 219 |
| 11.8. | Brownian Motion Model | 221 |
| | Summary of Chapter 11 | 223 |

| | | |
|-------------|---|-----|
| Chapter 12. | Circular Ensembles. Thermodynamics | 224 |
| 12.1. | The Partition Function | 224 |
| 12.2. | Thermodynamic Quantities | 227 |
| 12.3. | Statistical Interpretation of U and C | 229 |
| 12.4. | Continuum Model for the Spacing Distribution | 231 |
| | Summary of Chapter 12 | 236 |
| | | |
| Chapter 13. | Gaussian Ensemble of Anti-Symmetric Hermitian Matrices | 237 |
| 13.1. | Level Density. Correlation Functions | 237 |
| 13.2. | Level Spacings | 240 |
| 13.2.1. | Central Spacings | 240 |
| 13.2.2. | Non-Central Spacings | 242 |
| | Summary of Chapter 13 | 243 |
| | | |
| Chapter 14. | A Gaussian Ensemble of Hermitian Matrices With Unequal Real and Imaginary Parts | 244 |
| 14.1. | Summary of Results. Matrix Ensembles From GOE to GUE and Beyond | 245 |
| 14.2. | Matrix Ensembles From GSE to GUE and Beyond | 250 |
| 14.3. | Joint Probability Density for the Eigenvalues | 254 |
| 14.3.1. | Matrices From GOE to GUE and Beyond | 256 |
| 14.3.2. | Matrices From GSE to GUE and Beyond | 260 |
| 14.4. | Correlation and Cluster Functions | 263 |
| | Summary of Chapter 14 | 264 |
| | | |
| Chapter 15. | Matrices With Gaussian Element Densities But With No Unitary or Hermitian Conditions Imposed | 266 |
| 15.1. | Complex Matrices | 266 |
| 15.2. | Quaternion Matrices | 273 |
| 15.3. | Real Matrices | 279 |
| 15.4. | Determinants: Probability Densities | 281 |
| | Summary of Chapter 15 | 286 |
| | | |
| Chapter 16. | Statistical Analysis of a Level-Sequence | 287 |
| 16.1. | Linear Statistic or the Number Variance | 290 |
| 16.2. | Least Square Statistic | 294 |
| 16.3. | Energy Statistic | 298 |
| 16.4. | Covariance of Two Consecutive Spacings | 301 |
| 16.5. | The F-Statistic | 302 |
| 16.6. | The Λ -Statistic | 303 |
| 16.7. | Statistics Involving Three and Four Level Correlations | 303 |
| 16.8. | Other Statistics | 307 |
| | Summary of Chapter 16 | 308 |
| | | |
| Chapter 17. | Selberg's Integral and Its Consequences | 309 |
| 17.1. | Selberg's Integral | 309 |
| 17.2. | Selberg's Proof of Eq. (17.1.3) | 311 |

| | | |
|-------------|---|-----|
| 17.3. | Aomoto's Proof of Eqs. (17.1.4) and (17.1.3) | 315 |
| 17.4. | Other Averages | 318 |
| 17.5. | Other Forms of Selberg's Integral | 318 |
| 17.6. | Some Consequences of Selberg's Integral | 320 |
| 17.7. | Normalization Constant for the Circular Ensembles | 323 |
| 17.8. | Averages With Laguerre or Hermite Weights | 323 |
| 17.9. | Connection With Finite Reflection Groups | 325 |
| 17.10. | A Second Generalization of the Beta Integral | 327 |
| 17.11. | Some Related Difficult Integrals | 329 |
| | Summary to Chapter 17 | 334 |
| Chapter 18. | Asymptotic Behaviour of $E_\beta(0, s)$ by Inverse Scattering | 335 |
| 18.1. | Asymptotics of $\lambda_n(t)$ | 336 |
| 18.2. | Asymptotics of Toeplitz Determinants | 339 |
| 18.3. | Fredholm Determinants and the Inverse Scattering Theory | 340 |
| 18.4. | Application of the Gel'fand-Levitán Method | 342 |
| 18.5. | Application of the Marchenko Method | 347 |
| 18.6. | Asymptotic Expansions | 350 |
| | Summary of Chapter 18 | 353 |
| Chapter 19. | Matrix Ensembles and Classical Orthogonal Polynomials | 354 |
| 19.1. | Unitary Ensemble | 355 |
| 19.2. | Orthogonal Ensemble | 357 |
| 19.3. | Symplectic Ensemble | 361 |
| 19.4. | Ensembles With Other Weights | 363 |
| 19.5. | Conclusion | 363 |
| | Summary of Chapter 19 | 364 |
| Chapter 20. | Level Spacing Functions $E_\beta(r, s)$; Inter-relations and Power Series Expansions | 365 |
| 20.1. | Three Sets of Spacing Functions; Their Inter-Relations | 365 |
| 20.2. | Relation Between Odd and Even Solutions of Eq. (20.1.13) | 368 |
| 20.3. | Relation Between $F_1(z, s)$ and $F_\pm(z, s)$ | 371 |
| 20.4. | Relation Between $F_4(z, s)$ and $F_\pm(z, s)$ | 375 |
| 20.5. | Power Series Expansions of $E_\beta(r, s)$ | 376 |
| | Summary of Chapter 20 | 381 |
| Chapter 21. | Fredholm Determinants and Painlevé Equations | 382 |
| 21.1. | Introduction | 382 |
| 21.2. | Proof of Eqs. (21.1.11)–(21.1.17) | 385 |
| 21.3. | Differential Equations for the Functions A , B and S | 394 |
| 21.4. | Asymptotic Expansions for Large Positive τ | 396 |
| 21.5. | Fifth and Third Painlevé Transcendents | 400 |
| 21.6. | Solution of Eq. (21.3.6) for Large t | 406 |
| | Summary of Chapter 21 | 408 |

| | |
|--|-----|
| Chapter 22. Moments of the Characteristic Polynomial in the Three Ensembles of Random Matrices | 409 |
| 22.1. Introduction | 409 |
| 22.2. Calculation of $I_\beta(n, m; x)$ | 411 |
| 22.2.1. $I_\beta(n, m; x)$ as a determinant or a Pfaffian of a matrix of size depending on n | 412 |
| 22.2.2. $I_\beta(n, m; x)$ as determinants of size depending on m | 415 |
| 22.3. Special Case of the Gaussian Weight | 419 |
| 22.4. Average Value of $\prod_{i=1}^m \det(x_i I - A) \prod_{j=1}^\ell \det(z_j I - A)^{-1}$ | 421 |
| Summary of Chapter 22 | 424 |
| Chapter 23. Hermitian Matrices Coupled in a Chain | 426 |
| 23.1. General Correlation Function | 428 |
| 23.2. Proof of Theorem 23.1.1 | 430 |
| 23.3. Spacing Functions | 435 |
| 23.4. The Generating Function $R(z_1, I_1; \dots; z_p, I_p)$ | 437 |
| 23.5. The Zeros of the Bi-Orthogonal Polynomials | 441 |
| Summary of Chapter 23 | 448 |
| Chapter 24. Gaussian Ensembles. Edge of the Spectrum | 449 |
| 24.1. Level Density Near the Inflection Point | 450 |
| 24.2. Spacing Functions | 452 |
| 24.3. Differential Equations; Painlevé | 454 |
| Summary to Chapter 24 | 458 |
| Chapter 25. Random Permutations, Circular Unitary Ensemble (CUE) and Gaussian Unitary Ensemble (GUE) | 460 |
| 25.1. Longest Increasing Subsequences in Random Permutations | 460 |
| 25.2. Random Permutations and the Circular Unitary Ensemble | 461 |
| 25.3. Robinson–Schensted Correspondence | 463 |
| 25.4. Random Permutations and GUE | 468 |
| Summary of Chapter 25 | 468 |
| Chapter 26. Probability Densities of the Determinants; Gaussian Ensembles | 469 |
| 26.1. Introduction | 469 |
| 26.2. Gaussian Unitary Ensemble | 473 |
| 26.2.1. Mellin Transform of the PDD | 473 |
| 26.2.2. Inverse Mellin Transforms | 475 |
| 26.3. Gaussian Symplectic Ensemble | 477 |
| 26.4. Gaussian Orthogonal Ensemble | 480 |
| 26.5. Gaussian Orthogonal Ensemble. Case $n = 2m + 1$ Odd | 482 |
| 26.6. Gaussian Orthogonal Ensemble. Case $n = 2m$ Even | 483 |
| Summary of Chapter 26 | 486 |
| Chapter 27. Restricted Trace Ensembles | 487 |
| 27.1. Fixed Trace Ensemble; Equivalence of Moments | 487 |
| 27.2. Probability Density of the Determinant | 490 |