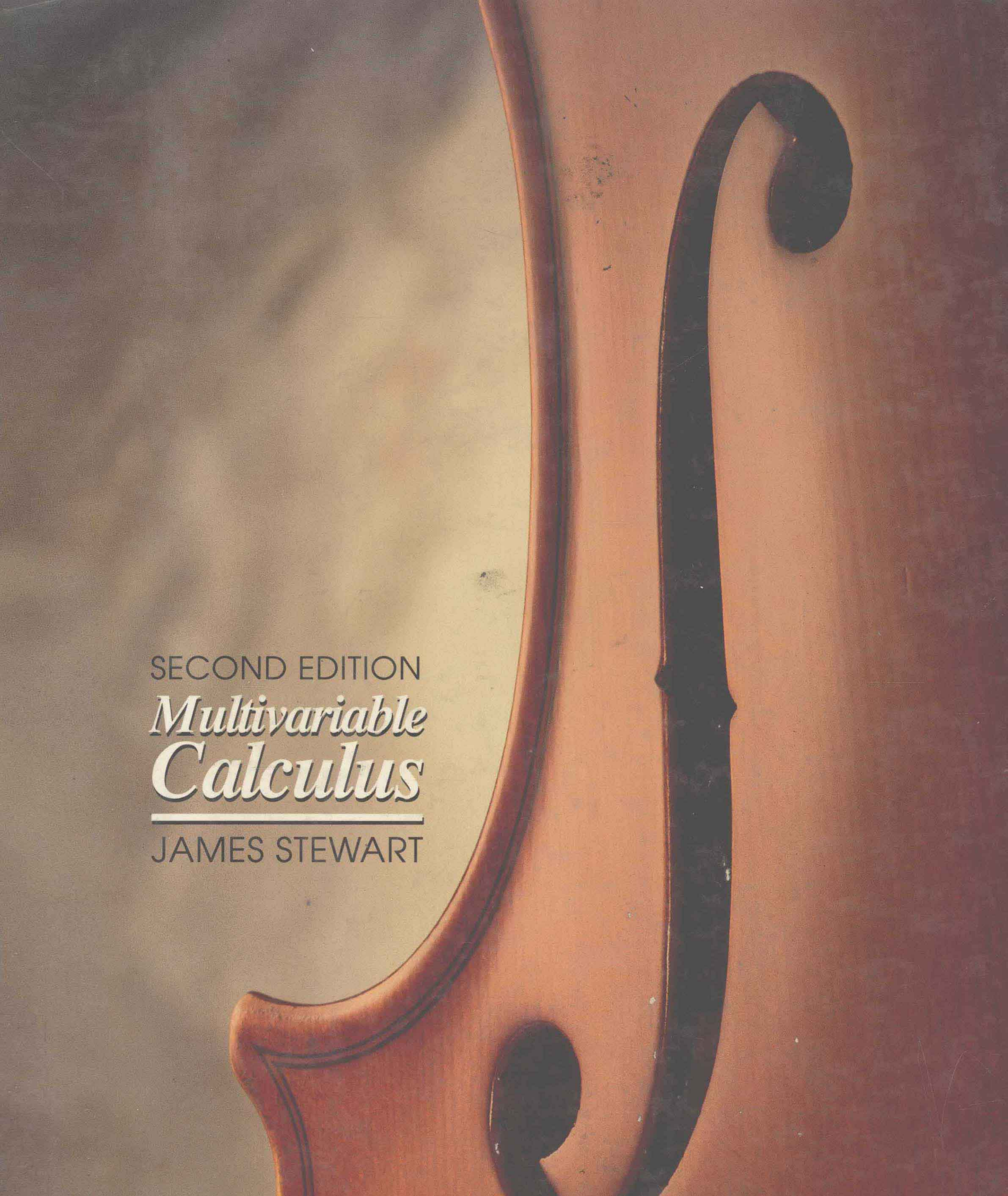

SECOND EDITION

Multivariable Calculus

JAMES STEWART

Multivariable Calculus

Second Edition

James Stewart

McMaster University

Brooks/Cole Publishing Company
Pacific Grove, California

To Vera, Sally, and Alan

Brooks/Cole Publishing Company
A Division of Wadsworth, Inc.
Multivariable Calculus, Second Edition, published in 1991, first appeared as Chapters 10–15 of *Calculus, Second Edition*, by James Stewart,

Printed in the United States of America
10 9 8 7 6 5 4 3 2

Library of Congress Cataloging-in-Publication Data
Stewart, James [date]
Multivariable calculus / James Stewart.—2nd ed.
p. cm.
Includes index.
ISBN 0-534-16344-0
1. Calculus. I. Title.
QA303.S88254 1991
515—dc20 91-26929
CIP

Sponsoring Editor: *Jeremy Hayhurst*
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Interior and Cover Design: *Katherine Minerva*
Cover Photo: *Lee Hocker*
Interior Illustration: *TECHarts*
Typesetting: *Jonathan Peck Typographers*
Cover Printing: *Lehigh Press Lithographers*
Printing and Binding: *Ringier America, New Berlin, WI*

Endpapers: From *Calculus with Analytic Geometry, Third Edition*, by E.W. Swokowski. Copyright © 1984 by PWS Publishers. Reprinted by permission.

Cover: Illustrates the remarkable resemblance between the sound hole of a violin viewed from this angle and the elongated "S" of the integral sign, a notation introduced by Leibniz.

Preface

This book is a reprinting of the multivariable portion of my text *Calculus, Second Edition*, published by Brooks/Cole in 1991. The chapters reproduced here cover: infinite sequences and series; three-dimensional analytic geometry and vectors; partial derivatives; multiple integrals; vector calculus; and differential equations. These chapters are a direct continuation of my *Single Variable Calculus, Second Edition*, also published in 1991, which contains Chapters 1–10 of *Calculus*. Particularly important results from single variable calculus are summarized for at-a-glance review in the section titled *Key Definitions, Properties, and Theorems from Single Variable Calculus*, following the table of contents.

While teaching from the first edition for three years, I (and my students) have had ideas, some major, some minor, for improving the exposition and organization and for adding new and better examples and exercises. I have also had the benefit of some valuable suggestions from colleagues, both friends and strangers, which have been incorporated into the second edition. Here is a summary of some of the principal changes:

■ In Chapter 10 there are more graphs of Taylor Series approximations, and Taylor's Formula is now proved in the text instead of in the exercises. Multiplication and division of power series are now covered.

■ Chapter 11 contains more applications of vectors in examples and exercises. Kepler's First Law is proved in the text although, as before, Laws 2 and 3 are left as exercises with hints.

■ Chapter 12 now contains many more computer graphics of surfaces and level curves, both in examples and exercises, and tree diagrams have been added to illustrate the Chain Rule. The geometric basis of Lagrange multipliers is explained.

■ A new section on surface area has been added in Chapter 13. Although parametric surfaces are still given a full treatment in Chapter 14, it is now possible to cover surface area and surface integrals nonparametrically.

■ A new review of complex numbers is included (page xxiv).

■ In this edition I have added what I call *Problems Plus* after even-numbered chapters. These are problems that go beyond the usual exercises in one way or another and require a higher level of problem-solving ability. The very fact that they do not occur in the context of any particular chapter makes them a little more challenging. I particularly value problems in which a student has to combine methods from two or three different chapters. In recent years

I have been testing these Problems Plus on my own students by putting them on assignments, tests, and exams. Because of their challenging nature I grade these problems in a different way. Here I reward a student significantly for ideas toward a solution and for recognizing which problem-solving principles are relevant. My aim is to teach my students to be unafraid to tackle a problem the likes of which they have never seen before.

■ A counterpart to the Problems Plus are the *Applications Plus*, which occur after odd-numbered chapters and which tend to be challenging because they involve related concepts from science that are usually outside students' experiences. Again the idea is to combine concepts and techniques from different parts of the book. These problems are helpful in demonstrating the sheer variety of the applications of calculus but also in focusing the students' attention on the essential mathematical similarities in diverse situations in science. By solving a wide variety of concrete problems, I hope that they will come to appreciate the power of abstraction. I am grateful to Garrett Etgen for amassing such a wide-ranging collection of applied problems.

I am indebted to the following astute reviewers, whose reasoned criticism enabled this book and its companion volumes to be better resources for both teacher and student:

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Tom Rishel, *Cornell University*
David Ryeburn, *Simon Fraser University*
Ricardo Salinas, *San Antonio College*
Stan Ver Nooy, *University of Oregon*
Jack Weiner, *University of Guelph*

In addition I would like to thank:

Harold Bacon and Mary Sunseri, *Stanford University*, who taught me how to teach calculus when I was a graduate student at Stanford.

Dan Anderson, *University of Iowa*, Dan Drucker, *Wayne State University*, and Barbara Frank, *The Pennsylvania State University*, for their careful working of the solutions to all the problems in the text and their authoring of Volumes 1 and 2 of the **Student Solutions Manual.** Student contributors were Aaron Childs, Loris Corazza, Michael Gmell, Brett Goodwin, and Tony di Silvestro.

David McKay, *California State University*, *Long Beach*, and Paul Yearout, professor emeritus from *Brigham Young University*, for proofing galleys and pages.

Bruce Char, *Oak Ridge National Laboratory*, *University of Tennessee*, David Hayes, *San Jose State University*, and Vincent Panico, *University of the Pacific*, for their helpful contributions to the appendix material.

Zdislav Kovarik, my colleague at *McMaster University*, for the ideas of Appendix D.

Garrett Etgen, *University of Houston*, for researching, writing, and solving the Applications Plus exercises, and Dan Drucker, *Wayne State University*, David McKay and Saleem Watson, *California State University, Long Beach*, and Lothar Redlin, *The Pennsylvania State University*, for checking the solutions.

Eric Bosch, *McMaster University*, and Dennis Schneider, *Knox College*, for their contributions of computer graphics.

Supplement Authors: Richard St. Andre, *Central Michigan University*, for the **Study Guide;** Joan Thomas and *Engineering Press, Inc.*, for their individual contributions to the **Test Item** file; J.S. Devitt, *University of Waterloo*, for his **Maple® and Calculus;** K. Heuvers, J. Kuisti, W. Francis, G. Ortner, D. Moak, and D. Lockhart, *Michigan Technological University*, for their **Linear Algebra for Calculus;** J. Douglas Child, *Rollins College*, Chris D'Arcy, *St. George's College*, Robert Kowalczyk and Adam Hausknecht, *Southeastern Massachusetts University*, Phoebe T. Judson, *Trinity University*, Vivian Y. Kraines, *Meredith College*, and Roy E. Myers, *The Pennsylvania State University*, for their individual contributions to **Calculus Laboratories for Brooks/Cole Software Tools.**

Most of these lab authors have also written the software that the labs require. Thanks to Ian Bell, Jon Davis, and Steve Rice, *Queen's University*, for **Calculus-Pad®**; Roy E. Myers, *The Pennsylvania State University*, for **Surface Plotter;** Chris D'Arcy, *St. George's College*, for **Image-Calculus;** J. Douglas Child, *Rollins College*, for **Calculus T/L;** Robert Kowalczyk and Adam Hausknecht, *Southeastern Massachusetts University*, for **TEMATH.**

Finally, I thank Kathi Townes for her production coordination and interior illustration for this text, Lee Hocker for the cover photograph, and the following Brooks/Cole staff: Joan Marsh, production services manager; Katherine Minerva, designer; Maureen Allaire, product manager; Margaret Parks, advertising manager; and Nancy Champlin, editorial assistant. Special thanks are due to the mathematics editor, Jeremy Hayhurst. He deserves much credit for the improvements in this edition.

JAMES STEWART

Key Definitions, Properties, and Theorems from *Single Variable Calculus*

Blue numbers refer to *Single Variable Calculus, Second Edition: Early Transcendentals*.

Definition (19)

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

Definition (1.3)

We write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a but not equal to a.

Limit Laws (1.6)

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

Further Properties of Limits

6. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer
    (If n is even, we assume that $a > 0$.)
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer
    (If n is even, we assume that $\lim_{x\to a} f(x) > 0$.)

The Squeeze Theorem (1.10)

If $f(x) \leqslant g(x) \leqslant h(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

Definition (1.12)

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L,** and we write

$$\lim_{x\to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - a| < \delta$$

Definition (1.16)

A function f is **continuous at a number a** if

$$\lim_{x\to a} f(x) = f(a)$$

Theorem (1.19)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

Theorem (1.24)

If g is continuous at a and f is continuous at $g(a)$, then $(f \circ g)(x) = f(g(x))$ is continuous at a.

Definition (2.2)

The **derivative of a function f at a number a,** denoted by $f'(a)$, is

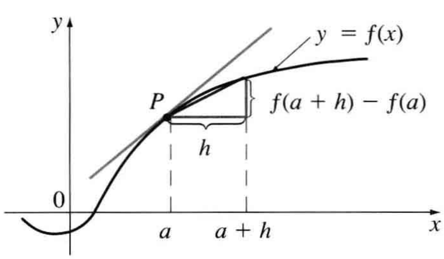
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

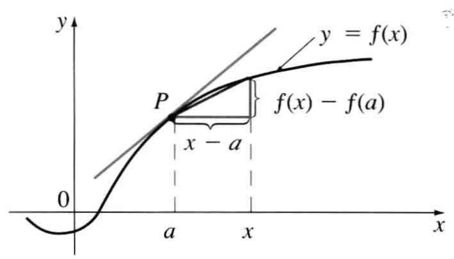
(2.3)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Geometric interpretation of the derivative



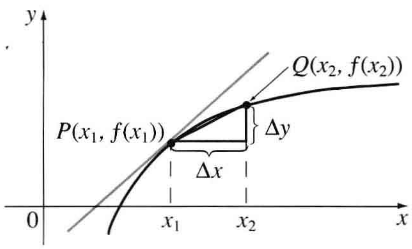
(a) $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ = slope of tangent at P



(b) $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ = slope of tangent at P

Theorem (2.8)

If f is differentiable at a, then f is continuous at a.



m_{PQ} = average rate of change
$m = f'(x_1)$ = instantaneous rate of change

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in the figure. Its limit as $\Delta x \to 0$ is the derivative $f'(x_1)$, which can therefore be interpreted as the instantaneous rate of change of y with respect to x or the slope of the tangent line at $P(x_1, f(x_1))$. Using the Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Table of Differentiation Formulas (2.17)

$$(cf)' = cf' \qquad (f + g)' = f' + g'$$

$$(f - g)' = f' - g' \qquad (fg)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \qquad \frac{d}{dx}\,c = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Theorem (2.22)

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

Table of Derivatives of Trigonometric Functions (2.29)

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

The Chain Rule (2.30)

If the derivatives $g'(x)$ and $f'(g(x))$ both exist, and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then $F'(x)$ exists and is given by the product

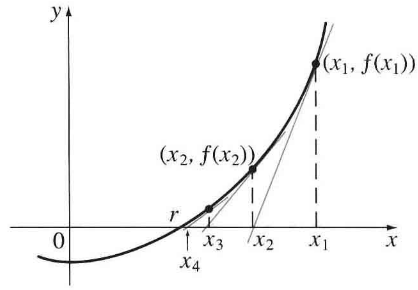
$$F'(x) = f'(g(x))g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Newton's Method (2.52)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Definition (3.1)
(4.1)

A function f has an **absolute maximum** at c if $f(c) \geqslant f(x)$ for all x in D, where D is the domain of f, and the number $f(c)$ is called the **maximum value** of f on D. Similarly, f has an **absolute minimum** at c if $f(c) \leqslant f(x)$ for all x in D and the number $f(c)$ is called the **minimum value** of f on D. The maximum and minimum values of f are called the **extreme values** of f.

Definition (3.2)
(4.2)

A function f has a **local maximum** (or **relative maximum**) at c if there is an open interval I containing c such that $f(c) \geqslant f(x)$ for all x in I. Similarly, f has a **local minimum** at c if there is an open interval I containing c such that $f(c) \leqslant f(x)$ for all x in I.

The Extreme Value Theorem (3.3)
(4.3)

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Fermat's Theorem (3.4)
(4.4)

If f has a local extremum (that is, maximum or minimum) at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Definition (3.6)
(4.6)

A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

(3.8)
(4.8)

To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of $f(a)$ and $f(b)$.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Rolle's Theorem (3.9)
(4.9)

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem (3.10)
(4.10)

Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

(3.11)
(4.11)

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

(3.12)
(4.12)

$$f(b) - f(a) = f'(c)(b - a)$$

Definition (3.18)
(4.18)

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **monotonic** on I if it is either increasing or decreasing on I.

Test for Monotonic Functions
(3.19)
(4.19)

Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

(a) If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.

The First Derivative Test (3.21)
(4.21)

Suppose that c is a critical number of a function f that is continuous on $[a, b]$.

(a) If $f'(x) > 0$ for $a < x < c$ and $f'(x) < 0$ for $c < x < b$ (that is, f' changes from positive to negative at c), then f has a local maximum at c.

(b) If $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$ (that is, f' changes from negative to positive at c), then f has a local minimum at c.

(c) If f' does not change sign at c, then f has no local extremum at c.

Definition (3.22)
(4.22)

If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of these tangents, it is called **concave downward** on I.

The Test for Concavity (3.23)
(4.23)

Suppose f is twice differentiable on an interval I.
(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

The Second Derivative Test (3.28)
(4.28)

Suppose f'' is continuous on an open interval that contains c.
(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Definition (3.29)
(1.26)

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Definition (3.33)
(1.30)

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad x > N$$

Definition (3.41)
(1.38)

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding number $N > 0$ such that

$$f(x) > M \qquad \text{whenever} \qquad x > N$$

Theorem (4.2)
(5.2)

If c is any constant (that is, it does not depend on i), then

(a) $\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i$

(b) $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

(c) $\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$

Definition of a Definite Integral (4.9)
(5.9)

If f is a function defined on a closed interval $[a, b]$, let P be a partition of $[a, b]$ with partition points $x_0, x_1, \ldots, x_n$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Choose points x_i^* in $[x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max\{\Delta x_i\}$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if this limit exists. If the limit does exist, then f is called **integrable** on the interval $[a, b]$.

Theorem (4.12)
(5.12)

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$$

Properties of the Integral (4.13)
(5.13)

Suppose that all of the following integrals exist. Then

1. $\int_a^b c\,dx = c(b - a)$, where c is any constant
2. $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Order Properties of the Integral (4.14) (5.14)

Suppose the following integrals exist and $a \leqslant b$.

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.
7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.
8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$$

9. $\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx$

The Fundamental Theorem of Calculus (4.29) (5.29)

Suppose f is continuous on $[a, b]$.

(a) If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

(b) $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

The Substitution Rule for Definite Integrals (4.34) (5.35)

If g' is continuous on $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Integrals of Symmetric Functions (4.35) (5.36)

Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^a f(x)\,dx = 0$.

(5.2) (6.2)

The area of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Definition of Volume (5.5) (6.5)

Let S be a solid that lies between the planes P_a and P_b. If the cross-sectional area of S in the plane P_x is $A(x)$, where A is an integrable function, then the **volume** of S is

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)\,dx$$

Mean Value Theorem for Integrals (5.17) (6.17)

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a)$$

(6.3) (3.3)

If $a > 1$, then

$$\lim_{x \to \infty} a^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} a^x = 0$$

If $0 < a < 1$, then

$$\lim_{x \to \infty} a^x = 0 \quad \text{and} \quad \lim_{x \to -\infty} a^x = \infty$$

Properties of Exponential Functions (6.9) (3.9)

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$$

$$\lim_{x \to \infty} \ln x = \infty \qquad \lim_{x \to 0^+} \ln x = -\infty$$

(6.25) (3.24)

$$\ln x = y \iff e^y = x$$

(6.26) (3.25)

$$\ln(e^x) = x \qquad x \in R$$

$$e^{\ln x} = x \qquad x > 0$$

(6.28) (3.27)

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

(6.36) (3.33)

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

(6.37)
(3.34)

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

Theorem (6.51)
(5.52)

$$\frac{d}{dx} e^x = e^x$$

Theorem (6.54)
(5.55)

$$\frac{d}{dx} a^x = a^x \ln a$$

Theorem (6.59)
(3.36)

The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

Table of Derivatives of Inverse Trigonometric Functions (6.72)
(3.49)

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Definition of the Hyperbolic Functions (6.76)
(3.50)

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

Table of Derivatives of Hyperbolic Functions (6.78)
(3.52)

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx}\coth x = -\operatorname{csch}^2 x$$