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Classical Descriptive Set Theory

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Preface

This book is based on some notes that I prepared for a class given at Caltech during the academic year 1991–92, attended by both undergraduate and graduate students. Although these notes underwent several revisions, which included the addition of a new chapter (Chapter V) and of many comments and references, the final form still retains the informal and somewhat compact style of the original version. So this book is best viewed as a set of lecture notes rather than as a detailed and scholarly monograph.

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Alexander S. Kechris

Introduction

Descriptive set theory is the study of "**definable sets**" in **Polish** (i.e., separable completely metrizable) **spaces**. In this theory, sets are classified in hierarchies, according to the complexity of their definitions, and the structure of the sets in each level of these hierarchies is systematically analyzed.

In the beginning we have the **Borel sets**, which are those obtained from the open sets, of a given Polish space, by the operations of complementation and countable union. Their class is denoted by $\mathbf{B}$. This class can be further analyzed in a transfinite hierarchy of length ω_1 (= the first uncountable ordinal), the **Borel hierarchy**, consisting of the open, closed, F_σ (countable unions of closed), G_δ (countable intersections of open), $F_{\sigma\delta}$ (countable intersections of F_σ), $G_{\delta\sigma}$ (countable unions of G_δ), etc., sets. In modern logical notation, these classes are denoted by $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, for $1 \leq \xi < \omega_1$, where

$$\begin{aligned}
\mathbf{\Sigma}^0_1 &= \text{ open, } \mathbf{\Pi}^0_1 = \text{ closed;}\\
\mathbf{\Sigma}^0_\xi &= \{\bigcup_{n\in\mathbb{N}} A_n : A_n \text{ is in } \mathbf{\Pi}^0_{\xi_n} \text{ for } \xi_n < \xi\};\\
\mathbf{\Pi}^0_\xi &= \text{ the complements of } \mathbf{\Sigma}^0_\xi \text{ sets.}
\end{aligned}$$

(Therefore, $\mathbf{\Sigma}^0_2 = F_\sigma$, $\mathbf{\Pi}^0_2 = G_\delta$, $\mathbf{\Sigma}^0_3 = G_{\delta\sigma}$, $\mathbf{\Pi}^0_3 = F_{\sigma\delta}$, etc.) Thus $\mathbf{B}$ ramifies in the following hierarchy:

$$\begin{array}{ccccccc}
\mathbf{\Sigma}^0_1 & \mathbf{\Sigma}^0_2 & & \mathbf{\Sigma}^0_\xi & & \mathbf{\Sigma}^0_\eta & \\
& & \cdots & & \cdots & & \cdots, \\
\mathbf{\Pi}^0_1 & \mathbf{\Pi}^0_2 & & \mathbf{\Pi}^0_\xi & & \mathbf{\Pi}^0_\eta &
\end{array}$$

where $\xi \leq \eta < \omega_1$, every class is contained in any class to the right of it, and

and applications in order to illustrate the general concepts and results of the theory. Moreover, over 400 exercises are included, of varying degrees of difficulty. Among them are important results as well as propositions and lemmas, whose proofs seem best to be left to the reader. A section at the end of these lectures contains hints to selected exercises.

This book is essentially self-contained. The only thing it requires is familiarity, at the beginning graduate or even advanced undergraduate level, with the basics of general topology, measure theory, and functional analysis, as well as the elements of set theory, including transfinite induction and ordinals. (See, for example, H. B. Enderton [1977], P. R. Halmos [1960a] or Y. N. Moschovakis [1994].) A short review of some standard set theoretic concepts and notation that we use is given in Appendices A and B. Appendix C explains some of the basic logical notation employed throughout the text. It is recommended that the reader become familiar with the contents of these appendices before reading the book and return to them as needed later on. On occasion, especially in some examples, applications, or exercises, we discuss material, drawn from various areas of mathematics, which does not fall under the preceding basic prerequisites. In such cases, it is hoped that a reader who has not studied these concepts before will at least attempt to get some idea of what is going on and perhaps look over a standard textbook in one of these areas to learn more about them. (If this becomes impossible, this material can be safely omitted.)

Finally, given the rather informal nature of these lectures, we have not attempted to provide detailed historical or bibliographical notes and references. The reader can consult the monographs by N. N. Lusin [1972], K. Kuratowski [1966], Y. N. Moschovakis [1980], as well as the collection by C. A. Rogers et al. [1980] in that respect. The Ω-*Bibliography of Mathematical Logic* (G. H. Müller, ed., Vol. 5, Springer-Verlag, Berlin, 1987) also contains an extensive bibliography.

About This Book

These lectures are divided into five chapters. The first chapter sets up the context by providing an overview of the basic theory of Polish spaces. Many standard tools, such as the Baire category theory, are also introduced here. The second chapter deals with the theory of Borel sets. Among other things, methods of infinite games figure prominently here, a feature that continues in the later chapters. In the third chapter, the theory of analytic sets, which is briefly introduced in the second chapter, is developed in more detail. The fourth chapter is devoted to the theory of co-analytic sets and, in particular, develops the machinery associated with ranks and scales. Finally, in the fifth chapter, we provide an introduction to the theory of projective sets, including the periodicity theorems.

We view this book as providing a first basic course in classical descriptive set theory, and we have therefore confined it largely to "core material" with which mathematicians interested in the subject for its own sake or those that wish to use it in their own field should be familiar. Throughout the book, however, are pointers to the literature for topics not treated here. In addition, a brief summary at the book's end (Section 40) describes the main further directions of current research in descriptive set theory.

Descriptive set theory can be approached from many different viewpoints. Over the years, researchers in diverse areas of mathematics—logic and set theory, analysis, topology, probability theory, and others—have brought their own intuitions, concepts, terminology, and notation to the subject. We have attempted in these lectures to present a largely balanced approach, which combines many elements of each tradition.

We have also made an effort to present a wide variety of examples

$$\mathbf{B} = \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi = \bigcup_{\xi<\omega_1} \mathbf{\Pi}^0_\xi.$$

Beyond the Borel sets one has next the **projective sets**, which are those obtained from the Borel sets by the operations of projection (or continuous image) and complementation. The class of projective sets, denoted by **P**, ramifies in an infinite hierarchy of length ω (= the first infinite ordinal), the **projective hierarchy**, consisting of the analytic (**A**) (continuous images of Borel), co-analytic (**CA**) (complements of analytic), **PCA** (continuous images of **CA**), **CPCA** (complements of **PCA**), etc., sets. Again, in logical notation, we let

$$\begin{aligned} \mathbf{\Sigma}^1_1 &= \text{ analytic, } \mathbf{\Pi}^1_1 = \text{ co-analytic;} \\ \mathbf{\Sigma}^1_{n+1} &= \text{ all continuous images of } \mathbf{\Pi}^1_n \text{ sets;} \\ \mathbf{\Pi}^1_{n+1} &= \text{ the complements of } \mathbf{\Sigma}^1_{n+1} \text{ sets;} \end{aligned}$$

so that in the following diagram every class is contained in any class to the right of it:

$$\begin{array}{cccccccc} & \mathbf{\Sigma}^1_1 & \mathbf{\Sigma}^1_2 & & \mathbf{\Sigma}^1_n & \mathbf{\Sigma}^1_{n+1} & & \\ \mathbf{B} & & & \cdots & & & \cdots \\ & \mathbf{\Pi}^1_1 & \mathbf{\Pi}^1_2 & & \mathbf{\Pi}^1_n & \mathbf{\Pi}^1_{n+1} & & \end{array}$$

and

$$\mathbf{P} = \bigcup_n \mathbf{\Sigma}^1_n = \bigcup_n \mathbf{\Pi}^1_n.$$

One can of course go beyond the projective hierarchy to study transfinite extensions of it, and even more complex "definable sets" in Polish spaces, but we will restrict ourselves here to the structure theory of Borel and projective sets, which is the subject matter of classical descriptive set theory.

Descriptive set theory has been one of the main areas of research in set theory for almost a century now. Moreover, its concepts and results are being used in diverse fields of mathematics, such as mathematical logic, combinatorics, topology, real and harmonic analysis, functional analysis, measure and probability theory, potential theory, ergodic theory, operator algebras, and topological groups and their representations. The main aim of these lectures is to provide a basic introduction to classical descriptive set theory and give some idea of its connections or applications to other areas.

Contents

CHAPTER I

Polish Spaces

1. Topological and Metric Spaces

1.A Topological Spaces

A **topological space** is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ a collection of subsets of X such that $\emptyset, X \in \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary unions and finite intersections. Such a collection is called a **topology** on X and its members **open** sets. The complements of open sets are called **closed**. Both $\emptyset, X$ are closed and arbitrary intersections and finite unions of closed sets are closed.

A set of the form $\bigcap_{n \in \mathbb{N}} U_n$, where U_n are open sets, is called a $\boldsymbol{G_\delta}$ set, and a set of the form $\bigcup_{n \in \mathbb{N}} F_n$, where F_n are closed sets, is called an $\boldsymbol{F_\sigma}$ set.

A **subspace** of $(X, \mathcal{T})$ consists of a subset $Y \subseteq X$ with the **relative topology** $\mathcal{T}|Y = \{U \cap Y : U \in \mathcal{T}\}$. (In general, for a set X, a subset $Y \subseteq X$, and a collection $\mathcal{A}$ of subsets of X, its **restriction** to Y is defined by $\mathcal{A}|Y = \{A \cap Y : A \in \mathcal{A}\}$.)

A **basis** $\mathcal{B}$ for a topology $\mathcal{T}$ is a collection $\mathcal{B} \subseteq \mathcal{T}$ with the property that every open set is the union of elements of $\mathcal{B}$. (By convention the empty union gives $\emptyset$.) For a collection $\mathcal{B}$ of subsets of a set X to be a basis for a topology, it is necessary and sufficient that the intersection of any two members of $\mathcal{B}$ can be written as a union of members of $\mathcal{B}$ and $\bigcup\{B : B \in \mathcal{B}\} = X$. A **subbasis** for a topology $\mathcal{T}$ is a collection $\mathcal{S} \subseteq \mathcal{T}$ such that the set of finite intersections of sets in $\mathcal{S}$ is a basis for $\mathcal{T}$. For any family $\mathcal{S}$ of subsets of a set X, there is a smallest topology $\mathcal{T}$ containing $\mathcal{S}$, called the **topology**

generated by $\mathcal{S}$. It consists of all unions of finite intersections of members of $\mathcal{S}$. (By convention the empty intersection gives X.) Clearly, $\mathcal{S}$ is a subbasis for $\mathcal{T}$. A topological space is **second countable** if it has a countable basis.

If X is a topological space and $x \in X$, an **open nbhd** (**neighborhood**) of x is an open set containing x. A **nbhd basis** for x is a collection $\mathcal{U}$ of open nbhds of x such that for every open nbhd V of x there is $U \in \mathcal{U}$ with $U \subseteq V$.

Given topological spaces X, Y, a map $f : X \to Y$ is **continuous** if the inverse image of each open set is open. It is **open** (resp. **closed**) if the image of each open (resp. closed) set is open (resp. closed). It is a **homeomorphism** if it is a bijection and is both continuous and open. Finally, it is called an **embedding** if it is a homeomorphism of X with $f(X)$ (given its relative topology). A function $f : X \to Y$ is **continuous at** $x \in X$ (or x is a **point of continuity** of f) if the inverse image of an open nbhd of $f(x)$ contains an open nbhd of x. So f is continuous iff it is continuous at every point.

If $(Y_i)_{i \in I}$ is a family of topological spaces and $f_i : X \to Y_i$, a family of functions, there is a smallest topology $\mathcal{T}$ on X for which all f_i are continuous. It is called the **topology generated by** $(f_i)_{i \in I}$ and has as a subbasis the family $\mathcal{S} = \{f_i^{-1}(U) : U \subseteq Y_i,\ U \text{ open},\ i \in I\}$. If $\mathcal{S}_i$ is a subbasis for the topology of Y_i, we can restrict U to $\mathcal{S}_i$ here.

The **product** $\prod_{i \in I} X_i$ of a family of topological spaces $(X_i)_{i \in I}$ is the topological space consisting of the cartesian product of the sets X_i with the topology generated by the projection functions $(x_i)_{i \in I} \mapsto x_j \ (j \in I)$. It has as basis the sets $\prod_i U_i$, where U_i is open in X_i for all $i \in I$, and $U_i = X_i$ for all but finitely many $i \in I$. If $\mathcal{B}_i$ is a basis for the topology of X_i, the sets of the form $\prod_i U_i$, where $U_i = X_i$ except for finitely many i for which $U_i \in \mathcal{B}_i$, form a basis for the product space. Note also that the projection functions are open. If $X_i = X$ for all $i \in I$, we let $X^I = \prod_{i \in I} X_i$.

The **sum** $\bigoplus_i X_i$ of a family of topological spaces $(X_i)_{i \in I}$ is defined (up to homeomorphism) as follows: If we replace X_i by a homeomorphic copy, we can assume that the sets X_i are pairwise disjoint. Let $X = \bigcup_{i \in I} X_i$. A set $U \subseteq X$ is open iff $U \cap X_i$ is open in X_i for each $i \in I$.

1.B Metric Spaces

A **metric space** is a pair (X, d), with X a set and $d : X^2 \to [0, \infty)$ a function satisfying:

i) $d(x, y) = 0 \Leftrightarrow x = y$;
ii) $d(x, y) = d(y, x)$;
iii) $d(x, y) \leq d(x, z) + d(z, y)$.

Such a function is called a **metric** on X.

The **open ball** with center x and radius r is defined by

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

(The corresponding **closed ball** is denoted by

$$B_{\text{cl}}(x,r) = \{y \in X : d(x,y) \leq r\}.)$$

These open balls form a basis for a topology, called the **topology of the metric space.**

A topological space $(X, \mathcal{T})$ is **metrizable** if there is a metric d on X so that $\mathcal{T}$ is the topology of (X, d). In this case we say that the metric d is **compatible** with $\mathcal{T}$. If $\mathcal{T}$ is metrizable with compatible metric d, then the metric

$$d' = \frac{d}{1+d}$$

is also compatible and $d' \leq 1$.

A subset $D \subseteq X$ of a topological space X is **dense** if it meets every nonempty open set. A space X admitting a countable dense set is called **separable**. Every second countable space is separable (but the converse does not hold). If X is metrizable, then X is separable iff X is second countable, so we use these terms interchangeably in this case.

A **subspace** of a metric space (X, d) is a subset $Y \subseteq X$ with the induced metric $d|Y$ (i.e., $d|Y(x, y) = d(x, y)$ for any $x, y \in Y$). The topology of $(Y, d|Y)$ is then the relative topology of Y. Thus a subspace of a metrizable topological space is metrizable. Moreover, a subspace of a separable metrizable space is separable.

A function $f : X \to Y$ between metric spaces (X, d_X), (Y, d_Y) is an **isometry** if it is a bijection and $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$. Every isometry is clearly a homeomorphism. We call f an **isometric embedding** if f is an isometry of X with $f(X)$.

The **product** of a sequence of metric spaces $\big((X_n, d_n)\big)_{n \in \mathbb{N}}$ is the metric space $(\prod_n X_n, d)$, where

$$d(x, y) = \sum_{n=0}^{\infty} 2^{-n-1} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)},$$

with $x = (x_n)$, $y = (y_n)$. The topology of this metric space is the product of the topologies of $\big((X_n, d_n)\big)$. Thus the product of a sequence of metrizable topological spaces is metrizable. Moreover, the product of a sequence of separable metrizable spaces is also separable. The **sum** of a family $\big((X_i, d_i)\big)_{i \in I}$ of metric spaces is defined (up to isometry) as follows: By copying the metric of each X_i on a set of the same cardinality, we can assume that the sets X_i are pairwise disjoint. Let $X = \bigcup_{i \in I} X_i$. We define a metric d on X by letting $d(x, y) = d_i(x, y)$, if $x, y \in X_i$, and $d(x, y) = 1$, if $x \in X_i$ and $y \in X_j$ with $i \neq j$. The topology of this metric space is the sum of the topologies of $\big((X_i, d_i)\big)$. Thus the sum of metrizable topological spaces is metrizable, and the sum of a sequence of separable metrizable spaces is separable.

We recall here the following important metrization theorem. A topological space X is called $\mathbf{T_1}$ if every singleton is closed and is called **regular**