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Eberhard Freitag
Rolf Busam

Complex Analysis

Second Edition

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Professor Dr. Eberhard Freitag
Universität Heidelberg
Inst. Mathematik
Im Neuenheimer Feld 288
69120 Heidelberg
Germany
freitag@mathi.uni-heidelberg.de

Dr. Rolf Busam
Universität Heidelberg
Inst. Mathematik
Im Neuenheimer Feld 288
69120 Heidelberg
Germany
busam@mathi.uni-heidelberg.de

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by Eberhard Freitag, Rolf Busam

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In Memoriam
Hans Maaß
(1911–1992)

Preface to the Second English Edition

Meanwhile the fourth edition of the German version appeared. In this second English edition we adapt the content closely to this German edition. We also followed several suggestions and tried to improve the language and typography. We thank Shari Scott for her support.

Heidelberg, August 2008

Eberhard Freitag
Rolf Busam

Preface to the English Edition

This book is a translation of the forthcoming fourth edition of our German book "Funktionentheorie I" (Springer 2005). The translation and the LaTeX files have been produced by Dan Fulea. He also made a lot of suggestions for improvement which influenced the English version of the book. It is a pleasure for us to express to him our thanks. We also want to thank our colleagues Diarmuid Crowley, Winfried Kohnen and Jörg Sixt for useful suggestions concerning the translation.

Over the years, a great number of students, friends, and colleagues have contributed many suggestions and have helped to detect errors and to clear the text.

The many new applications and exercises were completed in the last decade to also allow a partial parallel approach using computer algebra systems and graphic tools, which may have a fruitful, powerful impact especially in complex analysis.

Last but not least, we are indebted to Clemens Heine (Springer, Heidelberg), who revived our translation project initially started by Springer, New York, and brought it to its final stage.

Heidelberg, Easter 2005

Eberhard Freitag
Rolf Busam

Contents

Introduction

The complex numbers have their historical origin in the 16^{th} century when they were created during attempts to solve *algebraic equations*. G. CARDANO (1545) has already introduced formal expressions as for instance $5 \pm \sqrt{-15}$, in order to express solutions of quadratic and cubic equations. Around 1560 R. BOMBELLI computed systematically using such expressions and found 4 as a solution of the equation $x^3 = 15x + 4$ in the disguised form

$$4 = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \ .$$

Also in the work of G.W. LEIBNIZ (1675) one can find equations of this kind, e.g.

$$\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}} = \sqrt{6} \ .$$

In the year 1777 L. EULER introduced the notation $\mathrm{i} = \sqrt{-1}$ for the *imaginary* unit.

The terminology "complex number" is due to C.F. GAUSS (1831). The rigorous introduction of complex numbers as pairs of real numbers goes back to W.R. HAMILTON (1837).

Sometimes it is already advantageous to introduce and make use of complex numbers in real analysis. One should for example think of the integration of rational functions, which is based on the partial fraction decomposition, and therefore on the Fundamental Theorem of Algebra:

Over the field of complex numbers
any polynomial decomposes as a product of linear factors.

Another example for the fruitful use of complex numbers is related to FOURIER series. Following EULER (1748) one can combine the real angular functions sine and cosine, and obtain the "exponential function"

$$e^{\mathrm{i}x} := \cos x + \mathrm{i} \sin x \ .$$

Then the addition theorems for sine and cosine reduce to the simple formula

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$$e^{\mathrm{i}(x+y)} = e^{\mathrm{i}x}e^{\mathrm{i}y} \ .$$

In particular,

$$\left(e^{\mathrm{i}x}\right)^n = e^{\mathrm{i}nx} \text{ holds for all integers } n \ .$$

The FOURIER series of a sufficiently smooth function f, defined on the real line with period 1, can be written in terms of such expressions as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi \mathrm{i} n x} \ .$$

Here it is irrelevant whether f is real or complex valued.

In these examples the complex numbers serve as useful, but ultimately dispensable tools. New aspects come into play when we consider complex valued functions depending on a *complex variable*, that is when we start to study functions $f : D \to \mathbb{C}$ with two-dimensional domains D systematically. The dimension two is ensured when we restrict to *open domains of definition* $D \subset \mathbb{C}$. Analogously to the situation in real analysis one introduces the notion of complex differentiability by requiring the existence of the limit

$$f'(a) := \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

for all $a \in D$. It turns out that this notion has a drastically different behavior than real differentiability. We will show for instance that a (first order) complex differentiable function automatically is complex differentiable arbitrarily often. We will see more, namely that complex differentiable functions can always be developed locally as power series. For this reason, complex differentiable functions (defined on open domains) are also called *analytic functions*.

"Complex analysis" is the theory of such analytic functions.

Many classical functions from real analysis can be analytically extended to complex analysis. It turns out that these extensions are unique, as for instance in the case

$$e^{x+\mathrm{i}y} := e^x e^{\mathrm{i}y} \ .$$

It follows from the relation

$$\boxed{e^{2\pi \mathrm{i}} = 1}$$

that the complex exponential function is periodic with the *purely imaginary* period $2\pi\mathrm{i}$. This observation is fundamental for the complex analysis. As a consequence one can observe further phenomena:

1. The complex logarithm cannot be introduced as the unique inverse function of the exponential function in a natural way. It is *a priori* determined only up to a multiple of $2\pi\mathrm{i}$.

2. The function $1/z$ ($z \neq 0$) does not have any primitive in the punctured complex plane. A related fact is the following: the line integral of $1/z$ with respect to a circle line centered at the origin and oriented anticlockwise yields the non-zero value

$$\boxed{\oint_{|z|=r} \frac{1}{z}\,dz = 2\pi\mathrm{i} \qquad (r > 0)\,.}$$

Central results of complex analysis, like e.g. the *Residue Theorem*, are nothing but a highly generalized version of these statements.

Real functions often show their true nature only if one considers also their analytic extensions. For instance, in real analysis it is not directly transparent why the power series representation

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \pm \cdots$$

is valid only for $|x| < 1$. In the complex theory this phenomenon becomes understandable, simply because the considered function has singularities at $\pm\mathrm{i}$. Then its power series representation is valid in the biggest open disk excluding the singularities, namely the unit disk.

In real analysis it is also hard to understand why the TAYLOR series of the $\mathcal{C}^\infty$ function

$$f(x) = \begin{cases} e^{-1/x^2}\,, & x \neq 0\,, \\ 0\,, & x = 0\,, \end{cases}$$

around 0 converges for all $x \in \mathbb{R}$, but does not represent the function at any point other than zero. In the complex theory this phenomenon becomes understandable, since the function e^{-1/z^2} has an *essential singularity* at zero.

Less trivial examples are even more impressive. Here one should mention the RIEMANN ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\,,$$

which will be extensively studied in the last chapter of the book as a function of the *complex variable* traditionally denoted by s using the methods of complex analysis, which will be presented throughout the preceding chapters. From the analytical properties of the ζ-function we deduce the *Prime Number Theorem.*

RIEMANN's celebrated work on the ζ function [Ri2] is a brilliant example for the thesis he presented already eight years in advance in his dissertation [Ri1]

"Die Einführung der complexen Grössen in die Mathematik hat ihren Ursprung und nächsten Zweck in der Theorie einfacher durch Grössenoperationen ausgedrückter Abhängigkeitsgesetze zwischen veränderlichen Grössen. Wendet man nämlich diese Abhängigkeitsgesetze in einem erweiterten Umfange an, indem man den veränderlichen Grössen, auf welche sie sich beziehen, complexe Werthe giebt, so tritt eine sonst versteckt bleibende Harmonie und Regelmäßigkeit hervor."

In translation:

"The introduction of complex variables in mathematics has its origin and its proximate purpose in the theory of simple dependency rules for variables expressed by variable operations. If one applies these dependency rules in an extended manner by associating complex values to the variables referred to by these rules, then there emerges an otherwise hidden harmony and regularity."

Complex numbers are not only useful auxiliary tools, but in some respect are indispensable in many applications like e.g physics and other sciences: The commutation relations in quantum mechanics for impulse and coordinate operators

$$PQ - QP = \frac{h}{2\pi \mathrm{i}} I,$$

and respectively the SCHRÖDINGER equation for the HAMILTON operator H,

$$H\,\Psi(x,t) = \mathrm{i}\frac{h}{2\pi}\,\partial_t\Psi(x,t)$$

contain the imaginary unit i.

Since there exist several textbooks on complex analysis, a new attempt in this direction needed a special justification. The main idea of this book, and of a second forthcoming volume was to give an extensive description of classical complex analysis, where "classical" means that sheaf theoretical and cohomological methods are omitted. Obviously, it is not possible to include all material that can be considered as classical complex analysis. For instance, if somebody is especially interested in value distribution theory, or in applications of conformal maps, then one will be quickly disappointed and might put this book aside. The line pursued in this text can be described by keywords as follows:

The first four chapters contain an introduction to complex analysis, roughly corresponding to a course "complex analysis I" (four hours each week). Here, the fundamental results of complex analysis are treated.

After the foundations of the theory of analytic functions have been laid, we proceed to the theory of *elliptic functions*, then to *elliptic modular functions* —and after some excursions to analytic number theory— in a second volume we move on to *Riemann surfaces*, *the local theory of analytic functions of several variables*, to *abelian functions*, and finally we discuss *modular functions for several variables*.

Lot of emphasis is put on completeness of presentation, in the sense that all required notions and concepts are carefully developed. Except for basics in real analysis and linear algebra, as they are nowadays taught in standard introductory courses, we do not want to assume anything else in this first book. In a second volume only some simple topological concepts will be compiled without proof and subsequently used.

We made efforts to introduce as few notions as possible in order to quickly advance to the core of the studied problem. A series of important results will have several proofs. If a special case of a general proposition will be used in an important context, we strived to give a simpler proof for this special case as well. This is in accordance with our philosophy, that a thorough understanding can only be achieved if one turns things around and over and highlights them from different points of view.

We hope that this comprehensive presentation will convey a feeling for the way, in which the topics are related with each other, and for their origin.

Attempts like this are not new. Our text was primarily modelled on the lectures of H. MAASS, to whom we both owe our education in complex analysis. In the same breath, we would also like to mention the lectures of C.L. SIEGEL. Both sources are attempts to trace a great historical epoch, which among others is connected with the names of A.L. CAUCHY, N.H. ABEL, C.G.J. JACOBI, B. RIEMANN and K. WEIERSTRASS, and to present the results, which they developed.

Our objectives are very similar to both mentioned examples. Methodically however, our approach differs in many aspects. This will emerge especially in the second volume, where we will again dwell on the differences.

The present volume presents a comparatively simple introduction to the complex analysis in one variable. The content corresponds to a two semester course with accompanying seminars.

The first three chapters contain the standard material including the Residue Theorem, which should be covered in any introduction. In the fourth chapter —we rank it among the introductory lectures— we treat problems that are less obligatory. We present the Gamma function in detail to illustrate the methods by a beautiful example. We further focus on the Theorems of WEIERSTRASS and MITTAG-LEFFLER on the construction of analytic functions with prescribed zeros and poles. Finally, as a highlight, we prove the *Riemann Mapping Theorem* which states that any proper sub-domain of the complex plane $\mathbb{C}$ "without holes" is conform equivalent to the unit disk.

Only now, in an appendix to chapter IV we will treat the question of *simply connectedness* and we will give different equivalent characterizations for simply connected domains. Roughly speaking, they are domains without holes. In this context different versions, namely the homotopical and homological versions, of the CAUCHY Integral Theorem will be deduced.

These equivalences are useful for a good insight into the theory, and they are important for further developments. But nevertheless they are of minor importance for the development of the standard repertoire of complex analysis. Among simply connected domains we will only need *star-shaped domains* (and some domains that can be constructed from star-shaped domains). Consequently one needs the CAUCHY Integral Theorem only for star-shaped domains, which can be reduced to triangular paths by an idea of A. DINGHAS without any topological complications.

Therefore we will deliberately deal with star-shaped domains for a longer time to avoid the notion of simply connectedness. The price to be paid for this approach is that we have to introduce the concept of an *elementary domain*. By definition this is a domain where the CAUCHY Integral Theorem holds without exception. For this it is enough to know that star-shaped domains are elementary domains, and we postpone their final topological identification to the appendix of the fourth chapter. For the sake of a lucid methodology we have postponed this to a later point. In principle we could do without it completely in the first volume.

The subject of the fifth chapter is the theory of *elliptic functions*, i.e. meromorphic functions with two linearly independent periods. Historically these functions appeared as inverse functions of certain elliptic integrals, as for example the integral

$$\boxed{y = \int_*^x \frac{1}{\sqrt{1-t^4}}\, dt\ .}$$

It is easier to do the converse, namely to obtain the elliptic integrals as a byproduct of the impressively beautiful and simple theory of elliptic functions. One of the great achievements of complex analysis is the elegant and transparent construction of the theory of elliptic integrals. As quite usual nowadays, we will choose the WEIERSTRASS approach to the $\wp$-function.

In connection with ABEL's Theorem we will also give a short account of the historically older approach using the JACOBI theta function. We finish the fifth chapter by proving that any complex number is the absolute invariant of a period lattice. This fact implies that one indeed obtains any elliptic integral of the first kind as the inverse function of an elliptic function. At this point the elliptic modular function $j(\tau)$ appears.

As simple as this theory appears nowadays, it remains highly obscure how an elliptic integral gives rise to a period lattice, and thus to an elliptic function. In the second volume, the more complicated theory of RIEMANN surfaces will allow deeper insight into these questions.

In the sixth chapter we will systematically introduce —as a continuation of the end of fifth chapter— the theory of modular functions and modular forms.