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# Feynman's Operational Calculus and Beyond

## Noncommutativity and Time-Ordering

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and
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# DEDICATIONS

*To Gerald Johnson (Jerry), my wonderful coauthor, long-time friend and collaborator, for all his unique qualities, including his humor, incredible integrity, loyalty, generosity and perspicacity.*

*To his wife, Joan, his life-long companion and best friend, with gratitude for her friendship and for all she does for Jerry.*

*To my own wife and companion, Odile, the love of my life, without whom I cannot live.*

*To the memory of Richard Feynman and Mark Kac, friends and colleagues, with whom we shared many thoughts and joyful moments.*

*Michel L. Lapidus*

*I first met Jerry Johnson after asking some faculty at the University of Nebraska, Lincoln, if there was anyone in the department of mathematics who worked with path integrals. I was told to talk to Dr. Johnson, who was finishing class in such-and-such a room. I found this room and, while Jerry was erasing the chalk board, I asked if he knew of a good book concerning path integrals. The response was, with a smile, "I'm writing one." Of course, the book in question is that written by Jerry and Michel, i.e.,* The Feynman Integral and Feynman's Operational Calculus [*114*]. *This happenstance, some two decades ago, started me down the path that I continue to follow today.*

*As time moved on, Jerry consented to be my advisor, and I continue to be very thankful that I was lucky enough to have had someone with Jerry's wisdom, integrity, humor and patience as a mentor and friend during my time as a Ph.D. candidate and during my years in academia. It is because of Jerry that I learned how to do and write mathematics and, were it not for him, I would not be a coauthor of this volume. It has been a singular privilege to work on this volume with Jerry and also with Michel, both mathematicians of enormous talent and individuals I hold in the highest esteem.*

*Finally, I'd like to thank my dearest friend Amy for her good-humored tolerance to my continual protestations that "I can't, I have to work on the book." Amy and her youngest son Sam (as well as the family dog Lucky!) often gave me a needed escape from the manuscript when I could not bear to look at LaTeX any longer.*

*Lance Nielsen*

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Michel Lapidus

I would like to express my gratitude to the attendees—Jerry Johnson and Dave Skoug among others—of the functional integration seminar at the University of Nebraska, Lincoln, for the many opportunities to present and discuss my research, parts of which appear in this book. I would also like to thank the second author, Michel Lapidus, for providing a portion of the funding for a trip to Riverside, California, in 2009, to begin the process of writing this volume.

Lance Nielsen

# PREFACE

This book is aimed at providing a coherent, essentially self-contained, rigorous and comprehensive abstract theory of Feynman's operational calculus for noncommuting operators. Although it is inspired by Feynman's original heuristic suggestions and time-ordering rules in his seminal paper [58], as will be made abundantly clear in the introduction (Chapter 1) and elsewhere in the text, the theory developed in this book also goes well beyond them in a number of directions which were not anticipated in Feynman's work. Hence, the second part of the main title of this book.

It may be helpful to the reader to situate the present research monograph relative to a companion book [114], written by the first two named authors (Gerald Johnson and Michel Lapidus) and titled *The Feynman Integral and Feynman's Operational Calculus.* (Let us reassure the reader at once that [114] is *not* a prerequisite for the present book, however, as will be discussed in more detail further on in this preface.) The latter nearly 800-page book [114] was initially published in 2000 by Oxford University Press (with a paperback edition in 2002 and an electronic edition in the late 2000s) in the same series as the present monograph. It provides a number of different approaches to the Feynman path integral (or "sums over histories"), in both "real" and "imaginary" time.

Beginning with Chapter 14 and ending with Chapter 18, the second part of [114] (based, in part, on [110–113] along with [137–143]) develops a rigorous theory of Feynman's operational calculus, using certain operator-valued Wiener and Feynman path integrals (called "analytic-in-mass Feynman integrals") as well as associated commutative Banach algebras of functionals, called "disentangling algebras," and corresponding noncommutative operations (namely, a noncommutative addition and multiplication) acting on them. The resulting time-indexed family of disentangling algebras, along with the associated noncommutative operations, provides a rich algebraic, analytic and combinatorial structure for the development of a concrete theory of Feynman's operational calculus within the context of Feynman path integrals and related path or stochastic integrals.

On the other hand, Chapter 19 of [114] (based on the earlier joint work of the authors of [114] with Brian DeFacio in [33, 34]) begins to build a bridge between the above rigorous concrete version of the operational calculus and a possible, more general operational calculus valid for abstract operators (acting on Banach or Hilbert spaces) not necessarily arising via Wiener or Feynman functionals and associated path integrals. The connections with a large class of associated evolution equations are also studied in Chapter 19 of [114].

In a sense, Chapters 15–18 together with, specifically, Chapter 19 of [114] lay the foundations and provide a possible starting point for the development of a fully rigorous and more abstract theory of Feynman's operational calculus, which is the object of the present book. The reader familiar with Chapters 15–19 of [114] will recognize some aspects of, and motivations for, the theory developed in the present book, but in essence (with the notable exception of Chapter 19 of [114], which inherently serves as the basis for much of Chapter 6 of this monograph and is described in part in Section 6.2), the two theories and their presentations are essentially distinct and independent of one another. In particular, the present theory is aimed at dealing with abstract (typically) noncommuting operators, rather than operators arising from some kind of path integration (viewed as a suitable quantization procedure), as in [114]. In fact, some of the key structures developed in the present book (particularly, the family of commutative *disentangling algebras*, the corresponding *disentangling maps* and the associated *noncommutative operations*; see Chapters 2, 5 and 6) enable us, in some sense, to obtain an appropriate abstract substitute for a generalized functional integral (viewed as a suitable "quantization procedure" (in the sense of [143] and as described in [114, Section 18.6]) associated with the Feynman operational calculus attached to a given $n$-tuple of pairs $\{(A_j, \mu_j)\}_{j=1}^{n}$ of typically noncommuting bounded operators $A_j$ and probability measures $\mu_j$, for $j = 1, \ldots, n$ and $n \geq 2$).

As mentioned earlier, the present book is essentially self-contained. In particular, the earlier book [114] is not a prerequisite for understanding its contents. However, the interested reader may wish to consult Chapters 7 and 14 of [114], which provide a thorough introduction to the physical and heuristic aspects of "the" Feynman integral and Feynman's operational calculus, respectively, as well as to the associated and rather daunting mathematical difficulties. In the present book, we assume only that the reader has a reasonable graduate-level background in analysis, measure theory and functional analysis or operator theory.[1] Much of the necessary remaining background material is provided in the text itself.

In the introduction (Chapter 1) of this research monograph and elsewhere in the rest of the text (for example, in parts of Chapters 2, 3, 5, 6 and 8), we will present an overview of the heuristic and physical aspects of Feynman's operational calculus, with an eye towards the rigorous abstract theory developed in the book, based on time-ordering, noncommutativity, disentangling algebras, and associated disentangling maps and noncommutative operations. All of these notions will be progressively introduced and precisely defined, beginning with Chapter 2 and continuing on to Chapter 6, in particular. Along the way, several techniques for carrying out the "disentangling process," which is at the heart of Feynman's heuristic operator calculus proposed in [58], are developed throughout the book. See, for example, the discussion of the "disentangling of an exponential factor" (in Section 3.4 and, much more generally, in Chapter 6), the extraction of multilinear factors and iterative disentangling (in Chapter 4), the disentangling formulas (obtained in Chapter 5), the generalized Dyson expansions along with the corresponding evolution equations (in Chapter 6), the discussion of disentangling via the use of continuous and discrete measures (in Chapter 8), and the "derivation formulas" (via suitable functional derivatives in Chapter 9).

Reflecting upon the contents of this book, one sees in hindsight that the variety of disentangling techniques developed in the present theory constitutes one of its main features and

[1] See, for example, [13, 26, 41, 187, 192] for textbooks on these basic subjects; see also [11, 44, 78, 83, 123, 124, 188, 193, 195, 214] along with [114, Chapters 3, 6–10, 12 and 15] for more advanced material which will occasionally be needed in this book.

lies at the core of the present theory. We hope that the reader will find these disentangling results useful for his or her own purposes and will be stimulated to enrich the theory with new results, techniques and perspectives of an analytical, geometric, combinatorial or algebraic nature. The epilogue to this book (Chapter 11) has been written so as to facilitate this process and to suggest several possible directions for future research extending Feynman's operational calculus in a variety of ways.

Gerald W. Johnson, Michel L. Lapidus and Lance Nielsen
*March 2015*

# CONTENTS