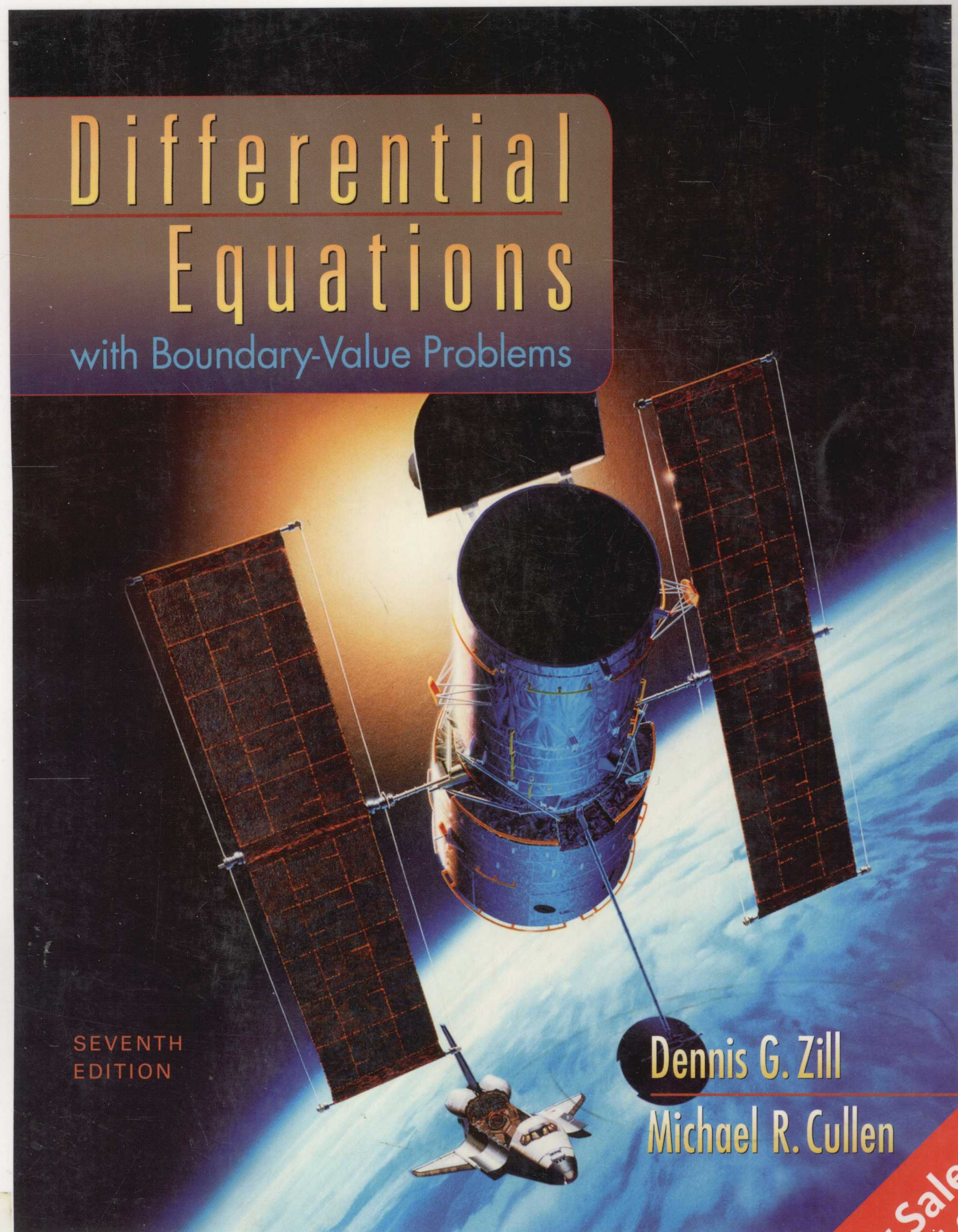
INTERNATIONAL STUDENT EDITION

Differential Equations

with Boundary-Value Problems



SEVENTH EDITION

Dennis G. Zill

Michael R. Cullen

Not for Sale in the United States

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DIFFERENTIAL EQUATIONS

with Boundary-Value Problems

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Dennis G. Zill and Michael R. Cullen

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PREFACE

TO THE STUDENT

Authors of books live with the hope that someone actually *reads* them. Contrary to what you might believe, almost everything in a typical college-level mathematics text is written for you and not the instructor. True, the topics covered in the text are chosen to appeal to instructors because they make the decision on whether to use it in their classes, but everything written in it is aimed directly at you the student. So I want to encourage you—no, actually I want to *tell* you—to read this textbook! But do not read this text like you would a novel; you should not read it fast and you should not skip anything. Think of it as a *work*book. By this I mean that mathematics should always be read with pencil and paper at the ready because, most likely, you will have to *work* your way through the examples and the discussion. Read—oops, work—all the examples in a section before attempting any of the exercises; the examples are constructed to illustrate what I consider the most important aspects of the section, and therefore, reflect the procedures necessary to work most of the problems in the exercise sets. I tell my students when reading an example, cover up the solution; try working it first, compare your work against the solution given, and then resolve any differences. I have tried to include most of the important steps in each example, but if something is not clear you should always try—and here is where the pencil and paper come in again—to fill in the details or missing steps. This may not be easy, but that is part of the learning process. The accumulation of facts followed by the slow assimilation of understanding simply cannot be achieved without a struggle.

Specifically for you, a *Student Resource and Solutions Manual* (*SRSM*) is available as an optional supplement. In addition to containing solutions of selected problems from the exercises sets, the *SRSM* has hints for solving problems, extra examples, and a review of those areas of algebra and calculus that I feel are particularly important to the successful study of differential equations. Bear in mind you do not have to purchase the *SRSM*; by following my pointers given at the beginning of most sections, you can review the appropriate mathematics from your old precalculus or calculus texts.

In conclusion, I wish you good luck and success. I hope you enjoy the text and the course you are about to embark on—as an undergraduate math major it was one of my favorites because I liked mathematics that connected with the physical world. If you have any comments, or if you find any errors as you read/work your way through the text, or if you come up with a good idea for improving either it or the *SRSM*, please feel free to either contact me or my editor at Brooks/Cole Publishing Company:

charlie.vanwagner@cengage.com

TO THE INSTRUCTOR

WHAT IS NEW IN THIS EDITION?

First, let me say what has *not* changed. The chapter lineup by topics, the number and order of sections within a chapter, and the basic underlying philosophy remain the same as in the previous editions.

In case you are examining this text for the first time, *Differential Equations with Boundary-Value Problems, 7th Edition,* can be used for either a one-semester course in ordinary differential equations, or a two-semester course covering ordinary and partial differential equations. The shorter version of the text, *A First Course in Differential Equations with Modeling Applications, 9th Edition*, ends with Chapter 9. For a one-semester course, I assume that the students have successfully completed at least two-semesters of calculus. Since you are reading this, undoubtedly you have already examined the table of contents for the topics that are covered. You will not find a "suggested syllabus" in this preface; I will not pretend to be so wise as to tell other teachers what to teach. I feel that there is plenty of material here to pick from and to form a course to your liking. The text strikes a reasonable balance between the analytical, qualitative, and quantitative approaches to the study of differential equations. As far as my "underlying philosophy" it is this: An undergraduate text should be written with the student's understanding kept firmly in mind, which means to me that the material should be presented in a straightforward, readable, and helpful manner, while keeping the level of theory consistent with the notion of a "first course."

For those who are familiar with the previous editions, I would like to mention a few of the improvements made in this edition.

- ***Contributed Problems*** Selected exercise sets conclude with one or two contributed problems. These problems were class tested and submitted by instructors of differential equations courses and reflect how they supplement their classroom presentations with additional projects.
- ***Exercises*** Many exercise sets have been updated by the addition of new problems to better test and challenge the students. In like manner, some exercise sets have been improved by sending some problems into early retirement.
- ***Design*** This edition has been upgraded to a four-color design, which adds depth of meaning to all of the graphics and emphasis to highlighted phrases. I oversaw the creation of each piece of art to ensure that it is as mathematically correct as the text.
- ***New Figure Numeration*** It took many editions to do so, but I finally became convinced that the old numeration of figures, theorems, and definitions had to be changed. In this revision I have utilized a double-decimal numeration system. By way of illustration, in the last edition Figure 7.52 only indicates that it is the 52nd figure in Chapter 7. In this edition, the same figure is renumbered as Figure 7.6.5, where

  Chapter Section
  ↓ ↓
  7.6.5 ← Fifth figure in the section

  I feel that this system provides a clearer indication to where things are, without the necessity of adding a cumbersome page number.
- ***Projects from Previous Editions*** Selected projects and essays from past editions of the textbook can now be found on the companion website at academic.cengage.com/math/zill.

STUDENT RESOURCES

- *Student Resource and Solutions Manual*, by Warren S. Wright, Dennis G. Zill, and Carol D. Wright (ISBN 0495385662 (accompanies A First Course in Differential Equations with Modeling Applications, 9e), 0495383163 (accompanies Differential Equations with Boundary-Value Problems, 7e)) provides reviews of important material from algebra and calculus, the solution of every third problem in each exercise set (with the exception of the Discussion Problems and Computer Lab Assignments), relevant command syntax for the computer algebra systems *Mathematica* and *Maple*, lists of important concepts, as well as helpful hints on how to start certain problems.

- *DE Tools* is a suite of simulations that provide an interactive, visual exploration of the concepts presented in this text. Visit academic.cengage.com/math/zill to find out more or contact your local sales representative to ask about options for bundling DE Tools with this textbook.

INSTRUCTOR RESOURCES

- *Complete Solutions Manual*, by Warren S. Wright and Carol D. Wright (ISBN 049538609X), provides worked-out solutions to all problems in the text.
- *Test Bank*, by Gilbert Lewis (ISBN 0495386065) Contains multiple-choice and short-answer test items that key directly to the text.

ACKNOWLEDGMENTS

Compiling a mathematics textbook such as this and making sure that its thousands of symbols and hundreds of equations are (mostly) accurate is an enormous task, but since I am called "the author" that is my job and responsibility. But many people besides myself have expended enormous amounts of time and energy in working towards its eventual publication. So I would like to take this opportunity to express my sincerest appreciation to everyone—most of them unknown to me—at Brooks/Cole Publishing Company, at Cengage Learning, and at Hearthside Publication Services who were involved in the publication of this new edition. I would, however, like to single out a few individuals for special recognition: At Brooks/Cole/Cengage, Cheryll Linthicum, Production Project Manager, for her willingness to listen to an author's ideas and patiently answering the author's many questions; Larry Didona for the excellent cover designs; Diane Beasley for the interior design; Vernon Boes for supervising all the art and design; Charlie Van Wagner, sponsoring editor; Stacy Green for coordinating all the supplements; Leslie Lahr, developmental editor, for her suggestions, support, and for obtaining and organizing the contributed problems; and at Hearthside Production Services, Anne Seitz, production editor, who once again put all the pieces of the puzzle together. Special thanks go to John Samons for the outstanding job he did reviewing the text and answer manuscript for accuracy.

I also extend my heartfelt appreciation to those individuals who took the time out of their busy academic schedules to submit a contributed problem:

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Doug Shaw, *University of Northern Iowa*
Warren S. Wright, *Loyola Marymount University*
David Zeigler, *California State University—Sacramento*

Finally, over the years these texts have been improved in a countless number of ways through the suggestions and criticisms of the reviewers. Thus it is fitting to conclude with an acknowledgement of my debt to the following people for sharing their expertise and experience.

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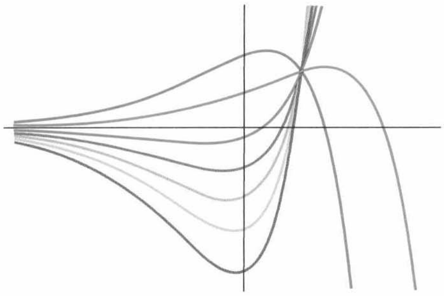
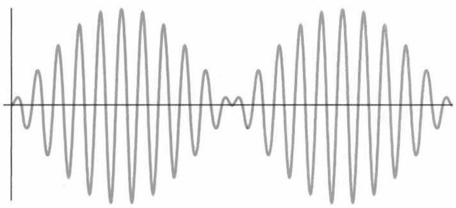
CONTENTS

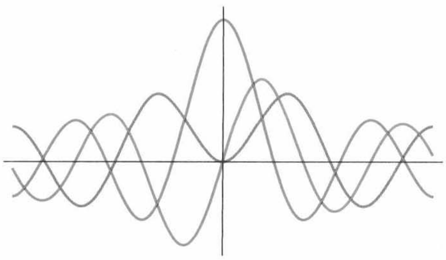
4 HIGHER-ORDER DIFFERENTIAL EQUATIONS 117



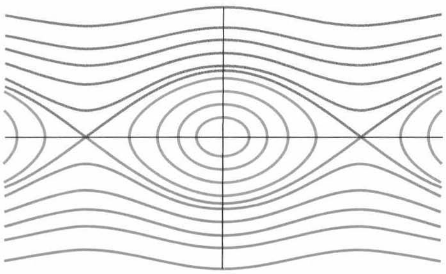
5 MODELING WITH HIGHER-ORDER DIFFERENTIAL EQUATIONS 181



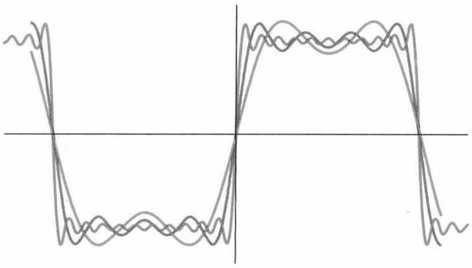
6 SERIES SOLUTIONS OF LINEAR EQUATIONS 219



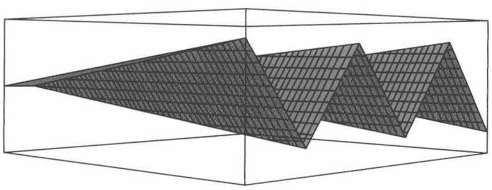
10 PLANE AUTONOMOUS SYSTEMS 363



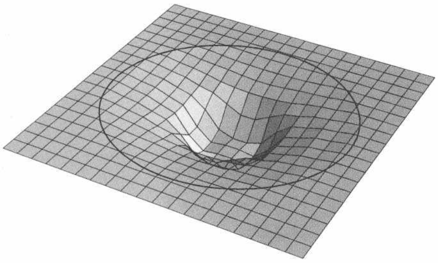
11 ORTHOGONAL FUNCTIONS AND FOURIER SERIES 397



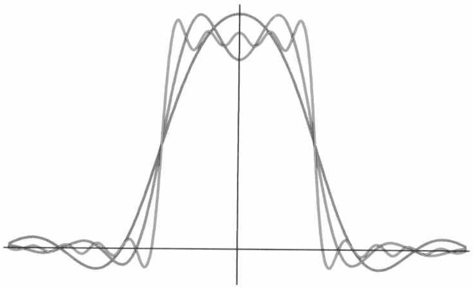
12 BOUNDARY-VALUE PROBLEMS IN RECTANGULAR COORDINATES 432



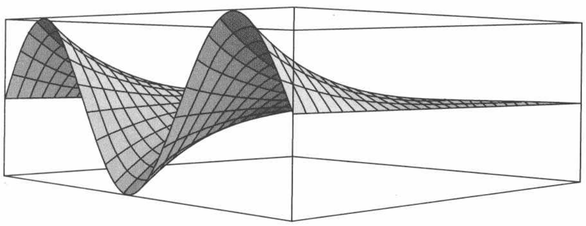
13 BOUNDARY-VALUE PROBLEMS IN OTHER COORDINATE SYSTEMS 471



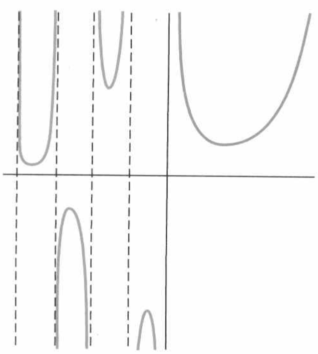
14 INTEGRAL TRANSFORMS 488



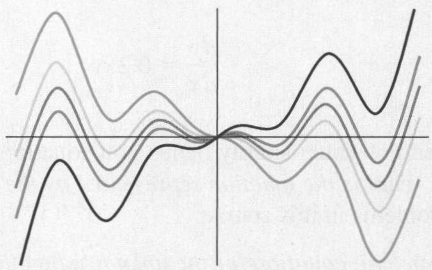
15 NUMERICAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS 511



APPENDICES



1 INTRODUCTION TO DIFFERENTIAL EQUATIONS



The words *differential* and *equations* certainly suggest solving some kind of equation that contains derivatives $y', y'', \ldots$. Analogous to a course in algebra and trigonometry, in which a good amount of time is spent solving equations such as $x^2 + 5x + 4 = 0$ for the unknown number x, in this course *one* of our tasks will be to solve differential equations such as $y'' + 2y' + y = 0$ for an unknown function $y = \phi(x)$.

The preceding paragraph tells something, but not the complete story, about the course you are about to begin. As the course unfolds, you will see that there is more to the study of differential equations than just mastering methods that someone has devised to solve them.

But first things first. In order to read, study, and be conversant in a specialized subject, you have to learn the terminology of that discipline. This is the thrust of the first two sections of this chapter. In the last section we briefly examine the link between differential equations and the real world. Practical questions such as *How fast does a disease spread? How fast does a population change?* involve rates of change or derivatives. As so the mathematical description—or mathematical model—of experiments, observations, or theories may be a differential equation.

1.1 DEFINITIONS AND TERMINOLOGY

REVIEW MATERIAL

- Definition of the derivative
- Rules of differentiation
- Derivative as a rate of change
- First derivative and increasing/decreasing
- Second derivative and concavity

INTRODUCTION The derivative dy/dx of a function $y = \phi(x)$ is itself another function $\phi'(x)$ found by an appropriate rule. The function $y = e^{0.1x^2}$ is differentiable on the interval $(-\infty, \infty)$, and by the Chain Rule its derivative is $dy/dx = 0.2xe^{0.1x^2}$. If we replace $e^{0.1x^2}$ on the right-hand side of the last equation by the symbol y, the derivative becomes

$$\frac{dy}{dx} = 0.2xy. \tag{1}$$

Now imagine that a friend of yours simply hands you equation (1)—you have no idea how it was constructed—and asks, *What is the function represented by the symbol* y? You are now face to face with one of the basic problems in this course:

How do you solve such an equation for the unknown function $y = \phi(x)$?

A DEFINITION The equation that we made up in (1) is called a **differential equation.** Before proceeding any further, let us consider a more precise definition of this concept.

DEFINITION 1.1.1 Differential Equation

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE).**

To talk about them, we shall classify differential equations by **type, order,** and **linearity.**

CLASSIFICATION BY TYPE If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an **ordinary differential equation (ODE).** For example,

A DE can contain more than one dependent variable

$$\frac{dy}{dx} + 5y = e^x, \qquad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \qquad \text{and} \qquad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y \tag{2}$$

are ordinary differential equations. An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a

partial differential equation (PDE). For example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3}$$

are partial differential equations.*

Throughout this text ordinary derivatives will be written by using either the **Leibniz notation** $dy/dx, d^2y/dx^2, d^3y/dx^3, \ldots$ or the **prime notation** $y', y'', y''', \ldots$. By using the latter notation, the first two differential equations in (2) can be written a little more compactly as $y' + 5y = e^x$ and $y'' - y' + 6y = 0$. Actually, the prime notation is used to denote only the first three derivatives; the fourth derivative is written $y^{(4)}$ instead of y''''. In general, the nth derivative of y is written d^ny/dx^n or $y^{(n)}$. Although less convenient to write and to typeset, the Leibniz notation has an advantage over the prime notation in that it clearly displays both the dependent and independent variables. For example, in the equation

unknown function
or dependent variable

$$\frac{d^2x}{dt^2} + 16x = 0$$

independent variable

it is immediately seen that the symbol x now represents a dependent variable, whereas the independent variable is t. You should also be aware that in physical sciences and engineering, Newton's **dot notation** (derogatively referred to by some as the "flyspeck" notation) is sometimes used to denote derivatives with respect to time t. Thus the differential equation $d^2s/dt^2 = -32$ becomes $\ddot{s} = -32$. Partial derivatives are often denoted by a **subscript notation** indicating the independent variables. For example, with the subscript notation the second equation in (3) becomes $u_{xx} = u_{tt} - 2u_t$.

CLASSIFICATION BY ORDER The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation. For example,

second order first order

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

is a second-order ordinary differential equation. First-order ordinary differential equations are occasionally written in differential form $M(x, y)\,dx + N(x, y)\,dy = 0$. For example, if we assume that y denotes the dependent variable in $(y - x)\,dx + 4x\,dy = 0$, then $y' = dy/dx$, so by dividing by the differential dx, we get the alternative form $4xy' + y = x$. See the *Remarks* at the end of this section.

In symbols we can express an nth-order ordinary differential equation in one dependent variable by the general form

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{4}$$

where F is a real-valued function of $n + 2$ variables: $x, y, y', \ldots, y^{(n)}$. For both practical and theoretical reasons we shall also make the assumption hereafter that it is possible to solve an ordinary differential equation in the form (4) uniquely for the

*Except for this introductory section, only ordinary differential equations are considered in *A First Course in Differential Equations with Modeling Applications,* Ninth Edition. In that text the word *equation* and the abbreviation DE refer only to ODEs. Partial differential equations or PDEs are considered in the expanded volume *Differential Equations with Boundary-Value Problems,* Seventh Edition.

highest derivative $y^{(n)}$ in terms of the remaining $n + 1$ variables. The differential equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}), \tag{5}$$

where f is a real-valued continuous function, is referred to as the **normal form** of (4). Thus when it suits our purposes, we shall use the normal forms

$$\frac{dy}{dx} = f(x, y) \qquad \text{and} \qquad \frac{d^2 y}{dx^2} = f(x, y, y')$$

to represent general first- and second-order ordinary differential equations. For example, the normal form of the first-order equation $4xy' + y = x$ is $y' = (x - y)/4x$; the normal form of the second-order equation $y'' - y' + 6y = 0$ is $y'' = y' - 6y$. See the *Remarks*.

CLASSIFICATION BY LINEARITY An nth-order ordinary differential equation (4) is said to be **linear** if F is linear in $y, y', \ldots, y^{(n)}$. This means that an nth-order ODE is linear when (4) is $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$ or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{6}$$

Two important special cases of (6) are linear first-order ($n = 1$) and linear second-order ($n = 2$) DEs:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad \text{and} \qquad a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{7}$$

In the additive combination on the left-hand side of equation (6) we see that the characteristic two properties of a linear ODE are as follows:

- The dependent variable y and all its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree, that is, the power of each term involving y is 1.
- The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ depend at most on the independent variable x.

The equations

$$(y - x)\,dx + 4x\,dy = 0, \qquad y'' - 2y' + y = 0, \qquad \text{and} \qquad \frac{d^3 y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

are, in turn, linear first-, second-, and third-order ordinary differential equations. We have just demonstrated that the first equation is linear in the variable y by writing it in the alternative form $4xy' + y = x$. A **nonlinear** ordinary differential equation is simply one that is not linear. Nonlinear functions of the dependent variable or its derivatives, such as $\sin y$ or $e^{y'}$, cannot appear in a linear equation. Therefore

$$\underbrace{(1 - y)y'}_{\substack{\text{nonlinear term:}\\ \text{coefficient depends on } y}} + 2y = e^x, \qquad \frac{d^2 y}{dx^2} + \underbrace{\sin y}_{\substack{\text{nonlinear term:}\\ \text{nonlinear function of } y}} = 0, \qquad \text{and} \qquad \frac{d^4 y}{dx^4} + \underbrace{y^2}_{\substack{\text{nonlinear term:}\\ \text{power not 1}}} = 0$$

are examples of nonlinear first-, second-, and fourth-order ordinary differential equations, respectively.

SOLUTIONS As was stated before, one of the goals in this course is to solve, or find solutions of, differential equations. In the next definition we consider the concept of a solution of an ordinary differential equation.