Fritz John

Applied Mathematical sciences 1

Partial Differential Equations

Fourth Edition

偏微分方程

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Preface to the fourth edition

A considerable amount of new material has been added to this edition. There is an extensive discussion of real analytic functions of several variables in Chapter 3. This permits estimation of the size of the domain of existence in the Cauchy–Kowalevski theorem. A first application of these estimates consists in a rigorous proof of a new version of Holmgren's uniqueness theorem for linear analytic partial differential equations (only sketched in the earlier editions). As another application (following Schauder) we give a second proof for existence of solutions of the initial value problem for symmetric hyperbolic systems in Chapter 5. Chapter 6 now includes a more detailed study of the Hilbert spaces $H_0^\mu(\Omega)$ with applications to the boundary behavior of solutions of the Dirichlet problem in higher dimensions. To Chapter 7 there has been added a proof of Widder's theorem on non-negative solutions of the heat equation. Finally, a new chapter, Chapter 8, contains H. Lewy's construction of a linear differential equation without solutions. There are also more problems, designed, in part, to extend the material discussed in the text.

I am particularly indebted to my colleague Percy A. Deift of the Courant Institute of New York University, to Prof. A. Garder of the Southern Illinois University at Edwardsville, Illinois, and to Dr. George Dassios of the National Technical University of Athens, Greece, for taking the trouble to compile lists of errors in the third edition. I hope that these have all been corrected and not too many new ones added in the present edition.

Beaverbrook,
Wilmington, New York

FRITZ JOHN

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1. Introduction

A partial differential equation (henceforth abbreviated as P.D.E.) for a function $u(x,y,\ldots)$ is a relation of the form

$$F(x,y,\ldots,u,u_x,u_y,\ldots,u_{xx},u_{xy},\ldots)=0, \tag{1.1}$$

where F is a given function of the independent variables $x,y,\ldots,$ and of the "unknown" function u and of a *finite* number of its partial derivatives. We call u a *solution* of (1.1) if after substitution of $u(x,y,\ldots)$ and its partial derivatives (1.1) is satisfied identically in $x,y,\ldots$ in some region Ω in the space of these independent variables. Unless the contrary is stated we require that $x,y,\ldots$ are real and that u and the derivatives of u occurring in (1.1) are continuous functions of $x,y,\ldots$ in the real domain Ω.† Several P.D.E.s involving one or more unknown functions and their derivatives constitute a *system*.

The *order* of a P.D.E. or of a system is the order of the highest derivative that occurs. A P.D.E. is said to be *linear* if it is linear in the unknown functions and their derivatives, with coefficients depending on the independent variables $x,y,\ldots$. The P.D.E. of order m is called *quasi-linear* if it is linear in the derivatives of order m with coefficients that depend on $x,y,\ldots$ and the derivatives of order $<m$.

*([6], [13], [27], [31])

†*For simplicity we shall often dispense with an explicit description of the domain* Ω. Statements made then apply "locally," in a suitably restricted neighborhood of a point of $xy\ldots$-space.

2. Examples

Partial differential equations occur throughout mathematics. In this section we give some examples. In many instances one of the independent variables is the time, usually denoted by t, while the others, denoted by $x_1, x_2, \ldots, x_n$ (or by x, y, z when $n \leq 3$) give position in an n-dimensional space. The space differentiations often occur in the particular combination

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{2.1}$$

known as the *Laplace operator*. This operator has the special property of being invariant under rigid motions or equivalently of not being affected by transitions to other cartesian coordinate systems. It occurs naturally in expressing physical laws that do not depend on a special position.

(i) The *Laplace equation* in n dimensions for a function $u(x_1, \ldots, x_n)$ is the linear second-order equation

$$\Delta u = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} = 0. \tag{2.2}$$

This is probably the most important individual P.D.E. with the widest range of applications. Solutions u are called *potential* functions or *harmonic* functions. For $n=2$, $x_1 = x$, $x_2 = y$, we can associate with a harmonic function $u(x,y)$ a "conjugate" harmonic function $v(x,y)$ such that the first-order system of *Cauchy–Riemann* equations

$$u_x = v_y, \qquad u_y = -v_x \tag{2.3}$$

is satisfied. A real solution (u, v) of (2.3) gives rise to the *analytic* function

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) \tag{2.4}$$

of the complex argument $z = x + iy$. We can also interpret $(u(x,y), -v(x,y))$ as the velocity field of an irrotational, incompressible flow. For $n=3$ equation (2.2) is satisfied by the velocity potential of an irrotational incompressible flow, by gravitational and electrostatic fields (outside the attracting masses or charges), and by temperatures in thermal equilibrium.

(ii) The *wave equation* in n dimensions for $u = u(x_1, \ldots, x_n, t)$ is

$$u_{tt} = c^2 \Delta u \tag{2.5}$$

($c = \text{const.} > 0$). It represents vibrations of strings or propagation of sound waves in tubes for $n=1$, waves on the surface of shallow water for $n=2$, acoustic or light waves for $n=3$.

(iii) *Maxwell's equations* in vacuum for the electric vector $E = (E_1, E_2, E_3)$ and magnetic vector $H = (H_1, H_2, H_3)$ form a linear system of essentially 6 first-order equations

$$\varepsilon E_t = \operatorname{curl} H, \qquad \mu H_t = -\operatorname{curl} E \tag{2.6a}$$

$$\operatorname{div} E = \operatorname{div} H = 0 \tag{2.6b}$$

with constants ε, μ. (If relations (2.6b) hold for $t=0$, they hold for all t as a consequence of relations (2.6a)). Here each component E_j, H_k satisfies the wave equation (2.5) with $c^2 = 1/\varepsilon\mu$.

(iv) *Elastic waves* are described classically by the linear system

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \mu \Delta u_i + (\lambda + \mu) \frac{\partial}{\partial x_i} (\operatorname{div} u) \tag{2.7}$$

($i = 1,2,3$), where the $u_i(x_1, x_2, x_3, t)$ are the components of the displacement vector u, and ρ is the density and λ, μ the Lamé constants of the elastic material. Each u_i satisfies the fourth-order equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \Delta \right) \left(\frac{\partial^2}{\partial t^2} - \frac{\mu}{\rho} \Delta \right) u_i = 0, \tag{2.8}$$

formed from two different wave operators. For *elastic* equilibrium ($u_t = 0$) we obtain the *biharmonic* equation

$$\Delta^2 u = 0. \tag{2.9}$$

(v) The equation of *heat conduction* ("heat equation")

$$u_t = k \Delta u \tag{2.10}$$

($k = \text{const.} > 0$) is satisfied by the temperature of a body conducting heat, when the density and specific heat are constant.

(vi) *Schrödinger's wave equation* ($n = 3$) for a single particle of mass m moving in a field of potential energy $V(x,y,z)$ is

$$i\hbar \psi_t = -\frac{\hbar^2}{2m} \Delta \psi + V\psi, \tag{2.11}$$

where $h = 2\pi\hbar$ is Planck's constant.

The equations in the preceding examples were all linear. Nonlinear equations occur just as frequently, but are inherently more difficult, hence in practice they are often approximated by linear ones. Some examples of nonlinear equations follow.

(vii) A *minimal surface* $z = u(x,y)$ (i.e., a surface having least area for a given contour) satisfies the second-order quasi-linear equation

$$(1 + u_y^2) u_{xx} - 2 u_x u_y u_{xy} + (1 + u_x^2) u_{yy} = 0. \tag{2.12}$$

(viii) The *velocity potential* $\phi(x,y)$ (for velocity components ϕ_x, ϕ_y) of a two-dimensional steady, adiabatic, irrotational, isentropic flow of density ρ satisfies

$$(1 - c^{-2}\phi_x^2)\phi_{xx} - 2c^{-2}\phi_x\phi_y\phi_{xy} + (1 - c^{-2}\phi_y^2)\phi_{yy} = 0, \tag{2.13}$$

where c is a known function of the speed $q = \sqrt{\phi_x^2 + \phi_y^2}$. For example

$$c^2 = 1 - \frac{\gamma - 1}{2} q^2 \tag{2.14}$$

for a polytropic gas with equation of state

$$p = A\rho^\gamma. \tag{2.15}$$

(ix) The *Navier–Stokes* equations for the viscous flow of an incompressible liquid connect the velocity components u_k and the pressure p:

$$\frac{\partial u_i}{\partial t} + \sum_k \frac{\partial u_i}{\partial x_k} u_k = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \gamma \Delta u_i, \tag{2.16a}$$

$$\sum_k \frac{\partial u_k}{\partial x_k} = 0, \tag{2.16b}$$

where ρ is the constant density and γ the kinematic viscosity.

(x) An example of a third-order nonlinear equation for a function $u(x,t)$ is furnished by the *Korteweg–de Vries* equation

$$u_t + cuu_x + u_{xxx} = 0 \tag{2.17}$$

first encountered in the study of water waves.

In general we shall try to describe the *manifold of solutions* of a P.D.E. The results differ widely for different classes of equations. Meaningful "well-posed" problems associated with a P.D.E. often are suggested by particular physical interpretations and applications.

3. Analytic Solution and Approximation Methods in a Simple Example*

We illustrate some of the notions that will play an important role in what follows by considering one of the simplest of all equations

$$u_t + cu_x = 0 \tag{3.1}$$

for a function $u = u(x,t)$, where $c = \text{const.} > 0$. Along a line of the family

$$x - ct = \text{const.} = \xi \tag{3.2}$$

("characteristic line" in the xt-plane) we have for a solution u of (3.1)

$$\frac{du}{dt} = \frac{d}{dt}u(ct+\xi,t) = cu_x + u_t = 0.$$

Hence u is constant along such a line, and depends only on the parameter ξ which distinguishes different lines. The general solution of (3.1) then has the form

$$u(x,t) = f(\xi) = f(x - ct). \tag{3.3}$$

Formula (3.3) represents the general solution u uniquely in terms of its *initial values*

$$u(x,0) = f(x). \tag{3.4}$$

Conversely every u of the form (3.3) is a solution of (3.1) with initial values f provided f is of class $C^1(\mathbf{R})$. We notice that the value of u at any point

*([18], [20], [25], [29])

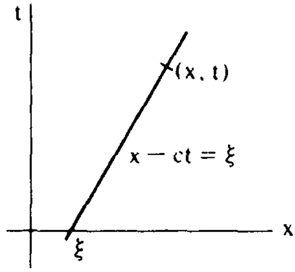

Figure 1.1

(x,t) depends only on the initial value f at the single argument $\xi = x - ct$, the abscissa of the point of intersection of the characteristic line through (x,t) with the initial line, the x-axis. The *domain of dependence* of $u(x,t)$ on the initial values is represented by the single point ξ. The *influence* of the initial values at a particular point ξ on the solution $u(x,t)$ is felt just in the points of the characteristic line (3.2). (Fig. 1.1)

If for each fixed t the function u is represented by its graph in the xu-plane, we find that the graph at the time $t = T$ is obtained by translating the graph at the time $t=0$ parallel to the x-axis by the amount cT:

$$u(x,0) = u(x+cT,T) = f(x).$$

The graph of the solution represents a *wave* propagating to the right with velocity c without changing shape. (Fig. 1.2)

We use this example with its explicit solution to bring out some of the notions connected with the numerical solution of a P.D.E by the *method of finite differences*. One covers the xt-plane by a rectangular grid with mesh size h in the x-direction and k in the t-direction. In other words one considers only points (x,t) for which x is a multiple of h and t a multiple of k. It would seem natural for purposes of numerical approximation to replace the P.D.E. (3.1) by the difference equation

$$\frac{v(x,t+k)-v(x,t)}{k} + c\frac{v(x+h,t)-v(x,t)}{h} = 0. \tag{3.5}$$

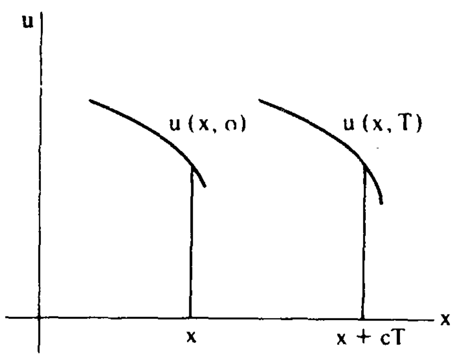

Figure 1.2

Formally this equation goes over into $v_t + cv_x = 0$ as $h,k \to 0$. We ask to what extent a solution v of (3.5) in the grid points with initial values

$$v(x,0) = f(x) \tag{3.6}$$

approximates for small h,k the solution of the initial-value problem (3.1), (3.4).

Setting $\lambda = k/h$, we write (3.5) as a recursion formula

$$v(x,t+k) = (1+\lambda c)v(x,t) - \lambda c v(x+h,t) \tag{3.7}$$

expressing v at the time $t+k$ in terms of v at the time t. Introducing the *shift operator E* defined by

$$Ef(x) = f(x+h), \tag{3.8}$$

(3.7) becomes

$$v(x,t+k) = ((1+\lambda c) - \lambda c E)v(x,t) \tag{3.8a}$$

for $t = nk$ this immediately leads by iteration to the solution of the initial-value problem for (3.5):

$$\begin{aligned} v(x,t) = v(x,nk) &= ((1+\lambda c) - \lambda c E)^n v(x,0) \\ &= \sum_{m=0}^{n} \binom{n}{m} (1+\lambda c)^m (-\lambda c E)^{n-m} f(x) \\ &= \sum_{m=0}^{n} \binom{n}{m} (1+\lambda c)^m (-\lambda c)^{n-m} f(x+(n-m)h). \end{aligned} \tag{3.9}$$

Clearly the domain of dependence for $v(x,t) = v(x,nk)$ consists of the set of points

$$x,\ x+h,\ x+2h,\ \ldots,\ x+nh \tag{3.10}$$

on the x-axis, all of which lie between x and $x+nh$. The domain of the differential equation solution consists of the point $\xi = x - ct = x - c\lambda nh$, which lies completely outside the interval $(x, x+nh)$. It is clear that v for $h,k \to 0$ cannot be expected to converge to the correct solution u of the differential equation, since in forming $v(x,t)$ we do not make use of any information on the value of $f(\xi)$, which is vital for determining $u(x,t)$, but only of more and more information on f in the interval $(x, x+(t/\lambda))$ which is irrelevant. The difference scheme fails the *Courant–Friedrichs–Lewy* test, which requires that the limit of the domain of dependence for the difference equation contains the domain of dependence for the differential equation.

That the scheme (3.5) is inappropriate also is indicated by its high degree of *instability*. In applied problems the data f are never known with perfect accuracy. Moreover, in numerical computations we cannot easily use the exact values but commit small round-off errors at every step. Now it is clear from (3.9) that errors in f of absolute value ε with the proper (alternating) sign can lead to a resulting error in $v(x,t)=v(x,nk)$ of size

$$\varepsilon\sum_{m=0}^{n}\binom{n}{m}(1+\lambda c)^{m}(\lambda c)^{n-m}=(1+2\lambda c)^{n}\varepsilon. \tag{3.11}$$

Thus for a fixed mesh ratio λ the possible resulting error in v grows exponentially with the number n of steps in the t-direction.

A more appropriate difference scheme uses "backward" difference quotients:

$$\frac{v(x,t+k)-v(x,t)}{k}+c\frac{v(x,t)-v(x-h,t)}{h}=0 \tag{3.12}$$

or symbolically

$$v(x,t+k)=((1-\lambda c)+\lambda cE^{-1})v(x,t). \tag{3.13}$$

The solution of the initial-value problem for (3.13) becomes

$$v(x,t)=v(x,nk)=\sum_{m=0}^{n}\binom{n}{m}(1-\lambda c)^{m}(\lambda c)^{n-m}f(x-(n-m)h). \tag{3.14}$$

In this scheme the domain of dependence for $v(x,t)$ on f consists of the points

$$x,\ x-h,\ x-2h,\ \ldots,\ x-nh=x-\frac{t}{\lambda} \tag{3.15}$$

Letting $h,k\to 0$ in such a way that the mesh ratio λ is held fixed, the set (3.15) has as its limit points the interval $[x-(t/\lambda),x]$ on the x-axis. The Courant–Friedrichs–Lewy test is satisfied, when this interval contains the point $\xi=x-ct$, that is when the mesh ratio λ satisfies

$$\lambda c\leq 1. \tag{3.16}$$

Stability of the scheme under the condition (3.16) is indicated by the fact that by (3.14) a maximum error of size ε in the initial function f results in a maximum possible error in the value of $v(x,t)=v(x,nk)$ of size

$$\varepsilon\sum_{m=0}^{n}\binom{n}{m}(1-\lambda c)^{m}(\lambda c)^{n-m}=\varepsilon((1-\lambda c)+\lambda c)^{n}=\varepsilon. \tag{3.17}$$

We can prove that the v represented by (3.14) actually converges to $u(x,t)=f(x-ct)$ for $h,k\to 0$ with $k/h=\lambda$ fixed, provided the stability criterion (3.16) holds and f has uniformly bounded second derivatives. For that purpose we observe that $u(x,t)$ satisfies

$$\begin{aligned}&|u(x,t+k)-(1-\lambda c)u(x,t)-\lambda cu(x-h,t)|\\&=|f(x-ct-ck)-(1-\lambda c)f(x-ct)-\lambda cf(x-ct-h)|\leq Kh^{2},\end{aligned} \tag{3.18}$$

where

$$K=\tfrac{1}{2}(c^{2}\lambda^{2}+\lambda c)\sup|f''|, \tag{3.19}$$

as is seen by expanding f about the point $x-ct$. Thus, setting $w=u-v$ we have

$$|w(x,t+k)-(1-\lambda c)w(x,t)-\lambda cw(x-h,t)| \leq Kh^2$$

and hence

$$\begin{aligned}\sup_x |w(x,t+k)| &\leq (1-\lambda c)\sup_x |w(x,t)| + \lambda c \sup_x |w(x-h,t)| + Kh^2 \\ &= \sup_x |w(x,t)| + Kh^2. \qquad (3.20)\end{aligned}$$

Applying (3.20) repeatedly it follows for $t=nk$ that

$$\begin{aligned}|u(x,t)-v(x,t)| &\leq \sup_x |w(x,nk)| \\ &\leq \sup_x |w(x,0)| + nKh^2 = \frac{Kth}{\lambda},\end{aligned}$$

since $w(x,0)=0$. Consequently $w(x,t)\to 0$ as $h\to 0$, that is, the solution v of the difference scheme (3.12) converges to the solution u of the differential equation.

PROBLEMS

1. Show that the solution v of (3.12) with initial data f converges to u for $h\to 0$ and a fixed $\lambda \leq 1/c$, under the sole assumption that f is continuous. (Hint: Use the fact that both u and v change by at most ε when we change f by at most ε.)

2. To take into account possible round-off errors we assume that instead of (3.13) v satisfies an inequality
$$|v(x,t+k)-(1-\lambda c)v(x,t)-\lambda cv(x-h,t)| < \delta.$$
Show that for a prescribed δ and for K given by (3.19) we have the estimate
$$|u(x,t)-v(x,t)| \leq \frac{Kth}{\lambda} + \frac{t}{\lambda h}\delta \qquad (3.21)$$
assuming that (3.16) holds and that $v(x,0)=f(x)$. Find values for λ and h based on this formula that will guarantee the smallest maximum error in computing $u(x,t)$.

3. Instability of a difference scheme under small perturbations does not exclude the possibility that in special cases the scheme converges towards the correct function, if no errors are permitted in the data or the computation. In particular let $f(x)=e^{\alpha x}$ with a complex constant α. Show that for fixed x, t and any fixed positive $\lambda = k/h$ whatsoever both the expressions (3.9) and (3.14) converge for $n\to\infty$ towards the correct limit $e^{\alpha(x-ct)}$. (This is consistent with the Courant–Friedrichs–Lewy test, since for an *analytic* f the values of f in any interval determine those at the point ξ uniquely.)