Jack K. Hale
Luis T. Magalhães
Waldyr M. Oliva

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Dynamics in Infinite Dimensions

Second Edition

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Appendix by Krzysztof P. Rybakowski

With 15 Figures

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Jack K. Hale
School of Mathematics
Georgia Tech
Atlanta, GA 30332-0001
USA
hale@math.gatech.edu

Luis T. Magalhães
Departamento de Matemática
Instituto Superior Técnico
Av. Rovisco Pais
Lisbon 1049-001
Portugal
lmagal@math.ist.utl.pt

Waldyr M. Oliva
Departamento de Matemática
Instituto Superior Técnico
Av. Rovisco Pais
Lisbon 1049-001
Portugal
wamoliva@math.ist.utl.pt

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Preface

In our book published in 1984 *An Introduction to Infinite Dimensional Dynamical Systems-Geometric Theory*, we presented some aspects of a geometric theory of infinite dimensional spaces with major emphasis on retarded functional differential equations. In this book, the intent is the same. There are new results on Morse–Smale systems for semiflows, persistence of hyperbolicity under perturbations, nonuniform hyperbolicity, monotone dynamical systems, realization of vector fields on center manifolds and normal forms. In addition, more attention is devoted to neutral functional differential equations although the theory is much less developed. Some parts of the theory also will apply to many other types of equations and applications.

Jack K. Hale
Luis T. Magalhães
Waldyr M. Oliva

Contents

1 Introduction

There is an extensive theory for the flow defined by dynamical systems generated by continuous semigroups $T : \mathbb{R}^+ \times \mathcal{M} \to \mathcal{M}$, $T(t,x) := T(t)x$, where $T(t) : \mathcal{M} \to \mathcal{M}$, $\mathbb{R}^+ = [0, \infty)$, and $\mathcal{M}$ is either a finite dimensional compact manifold without boundary or a compact manifold with boundary provided that the flow is differentiable and transversal to the boundary. The basic problem is to compare the flows defined by different dynamical systems. This comparison is made most often through the notion of topological equivalence. Two semigroups T and S defined on $\mathcal{M}$ are *topologically equivalent* if there is a homeomorphism from $\mathcal{M}$ to $\mathcal{M}$ which takes the orbits of T onto the orbits of S and preserves the sense of direction in time.

If the semigroups are defined on a finite dimensional Banach space X, then extreme care must be exercised in order to compare the orbits with large initial data and only very special cases have been considered. One way to avoid the consideration of large initial data in the comparison of semigroups is to consider only those semigroups for which infinity is unstable; that is, there is a bounded set which attracts the *positive orbit* of each point in X. In this case, there is a *compact global attractor* $\mathcal{A}(T)$ of the semigroup T; that is, $\mathcal{A}(T)$ is *compact invariant* ($T(t)\mathcal{A}(T) = \mathcal{A}(T)$ for all $t \geq 0$) and, in addition, for any bounded set $B \subset X$, $\text{dist}_X(T(t)B, \mathcal{A}(T)) \to 0$ as $t \to \infty$. In such situations, it is often possible to find a neighborhood $\mathcal{M}$ of $\mathcal{A}(T)$ for which the closure is a compact manifold with boundary and the boundary is transversal to the flow. Therefore, the global theory of finite dimensional dynamical systems can be applied.

We remark that the invariance of $\mathcal{A}(T)$ implies that, for each $x \in \mathcal{A}(T)$, we can define a bounded *negative orbit* (or a bounded *backward extension*) through x; that is, a function $\varphi : (-\infty, 0] \to X$ such that $\varphi(0) = x$ and, for any $\tau \leq 0$, $T(t)\varphi(\tau) = \varphi(t + \tau)$ for $0 \leq t \leq -\tau$. If the compact global attractor $\mathcal{A}(T)$ exists, then it is given by

$$\mathcal{A}(T) = \{x \in X : T(t)x \text{ is defined and bounded for } t \in \mathbb{R}\}. \tag{1.1}$$

Many applications involve semigroups T on a non-locally compact space X; for example, semigroups generated by partial differential equations and delay differential or functional differential equations (see for example [8], [78], [86], [198] and the references therein). The first difficulty in the non-locally compact case is to decide how to compare two semigroups. It seems to be almost impossible to make a comparison of all or even an arbitrary bounded set of the space X. On the other hand, we can define in the non-locally compact case the set $\mathcal{A}(T)$ as in (1.1). This set will contain all of the bounded invariant sets of T and, under some reasonable conditions, should contain all of the information about the limiting behavior of solutions. For this reason, we make comparisons of semigroups only on $\mathcal{A}(T)$. This does not mean that the transient behavior is unimportant, but only that our emphasis here is on $\mathcal{A}(T)$. The following definition first appeared in a paper by Hale (see [73]) in 1981.

Definition 1.0.1. *We say that a semigroup T on X is equivalent to a semigroup S on X, $T \sim S$, if there is a homeomorphism $h : \mathcal{A}(T) \to \mathcal{A}(S)$ which preserves orbits and the sense of direction in time.*

We reemphasize that, in the definition of equivalence, we restrict to the set $\mathcal{A}(T)$ and not to a neighborhood of $\mathcal{A}(T)$. Due to the fact that we are not able to take this full neighborhood, adaptation of the finite dimension theory of dynamical systems to our setting is nontrivial. Also, we will need to impose further restrictions on the classes of semigroups that will be considered.

If $\mathcal{A}(T)$ is not compact, there is very little known about general flows. If $\mathcal{A}(T)$ is compact, then we can easily verify the following result.

Proposition 1.0.2. *If $\mathcal{A}(T)$ is compact, then $\mathcal{A}(T)$ is the maximal compact invariant set. If, in addition, for each $t \geq 0$, $T(t)$ is one-to-one on $\mathcal{A}(T)$, then T is a continuous group on $\mathcal{A}(T)$.*

In a particular application, the semigroup defining the dynamical system depends upon parameters. In the case of ordinary differential equations or functional differential equations, the parameter could be a particular class of vector fields. If the semigroup is generated by partial differential equations, the parameters could be a class of vector fields or the boundary of the region of definition or the boundary conditions or all of these. A basic problem is to know if the flow defined by the a semigroup is preserved under the above equivalence relation when one allows variations in the parameters. More precisely, we make the following definitions.

Definition 1.0.3. *Suppose that X is a complete metric space, Λ is a metric space, $T : \Lambda \times \mathbb{R}^+ \times X \to X$ is continuous and, for each $\lambda \in \Lambda$, let $T_\lambda : \mathbb{R}^+ \times X \to X$ be defined by $T_\lambda(t)x = T(\lambda, t, x)$ and suppose that T_λ is a continuous semigroup on X for each $\lambda \in \Lambda$. Define $\mathcal{A}(T_\lambda)$ as above. The semigroup T_λ is said to be $\mathcal{A}$-stable if there is a neighborhood $U \subset \Lambda$ of λ such that $T_\lambda \sim T_\mu$ for each $\mu \in U$. We say that T_λ is a bifurcation point if T_λ is not $\mathcal{A}$-stable.*

The basic problem is to discuss detailed properties of the set $\mathcal{A}(T_\lambda)$, the structure of the flow on $\mathcal{A}(T_\lambda)$ and the manner in which $\mathcal{A}(T_\lambda)$ changes with λ.

Some basic questions that should be discussed are the following:

1. Is T_λ generically one-to-one on $\mathcal{A}(T_\lambda)$?
2. If T_λ is $\mathcal{A}$-stable, is T_λ one-to-one on $\mathcal{A}(T_\lambda)$?
3. For each $x \in \mathcal{A}(T_\lambda)$, what are the smoothness properties of $T_\lambda(t)x$ in t? For example, does it possess the same smoothness properties as the semigroup has in x and or λ?
4. Is the Hausdorff dimension and capacity of $\mathcal{A}(T_\lambda)$ finite?
5. When is $\mathcal{A}(T_\lambda)$ a manifold or the union of a finite number of manifolds?
6. Can $\mathcal{A}(T_\lambda)$ be embedded in a finite dimensional manifold generically in λ?
7. Can $\mathcal{A}(T_\lambda)$ be embedded in a finite dimensional invariant manifold generically in λ?
8. Are Morse–Smale systems open and $\mathcal{A}$-stable?
9. Are Kupka–Smale semigroups generic in the class $\{T_\lambda, \lambda \in \Lambda\}$?

In these notes, we attempt to discuss these questions in some detail in order to indicate how one can begin to obtain a geometric theory for dynamical systems in infinite dimensions. We present some results which apply to many types of situations including functional differential equations of retarded and neutral type, quasilinear parabolic partial differential equations and dissipative hyperbolic partial differential equations. Some of the more detailed results are for a class of semigroups satisfying compactness and smoothness hypotheses and are directly applicable to retarded functional differential equations with finite delay, quasilinear parabolic equations and more general situations. There are many important applications which do not satisfy the compactness and smoothness hypotheses; for example, retarded equations with infinite delay, neutral functional differential equations, the linearly damped nonlinear wave equation as well as other equations of hyperbolic type. Throughout, we will note the difficulties involved in the extensions to more general semigroups. As will be clear, the theory is still in its infancy.

Our theory is presented for the case in which $\mathcal{A}(T)$ is compact. In Chapter 2, we present conditions on the semigroup T and dissipative properties of the flow which will imply that $\mathcal{A}(T)$ is compact and, therefore, is the maximal compact invariant set. Also, we give conditions which are necessary and sufficient for $\mathcal{A}(T)$ to be the compact global attractor.

In Chapter 3, we give the definitions and examples of retarded and neutral functional differential equations on manifolds, discuss the basic properties of the semigroups defined by these equations, the existence of compact global attractors and the differences between these two types of equations.

In Chapter 4, we show that the compact global attractor has finite capacity for a class of mappings which includes the time-one maps of retarded and neutral functional differential equations, linearly damped hyperbolic equations as well as many other types of equations.

In Chapter 5, we give some examples illustrating the importance of discussing the manner in which the flow on the compact global attractor depends upon parameters. A rather complete investigation is made for a retarded functional differential equation serving as a model in viscoelasticity and known as the Levin-Nohel equation with the parameter being the relaxation function. Also, a complete description is given for the flow on the attractor for a scalar parabolic equation in one dimension. Some details in the proof are referred to Chapter 10. A counter-example for the Hartman-Grobman theorem in the setting of Hadamard derivatives is also described.

In Chapter 6, the definitions of Morse–Smale maps and flows are given. The stability of Morse–Smale maps was proved in [87] and is reproduced here. The stability for semiflows is a recent result appearing in [154] and, as will be seen, there are some conditions imposed on the flow which involve smoothness. These smoothness conditions are satisfied for retarded functional differential equations and parabolic equations, but are not satisfied for neutral functional differential equations and partial differential equations for which the solutions do not smooth in time. It would be very interesting to extend the results in this chapter to more general situations.

Chapter 7 is devoted to the persistence under perturbations for uniformly hyperbolic invariant sets of semiflows, assuming the smoothness condition mentioned above. The hypothesis that the flow is one-to-one on a compact invariant set implies the existence of a conjugacy between perturbed and unperturbed semiflows; if the flow on the invariant set is not one-to-one, one obtains only a semi-conjugacy. Hyperbolic measures and nonuniform hyperbolicity together with the corresponding concepts of invariant manifolds are discussed in the finite dimensional case with some remarks on perspectives for the infinite dimensional setting.

Even though the flow defined by evolutionary equations defined by functional differential equations and partial differential equations are defined on an infinite dimensional space, the particular type of equation considered may impose restrictions on the flow. This can play a very important role in the development of the geometric theory and have an important impact on the types of bifurcations that may occur. In Chapter 8, we characterize the flows that can occur on center manifolds for retarded and neutral functional differential equations. This chapter also contains a complete theory of normal forms for these equations as well as abstract evolutionary equations with delays with applications.

In Chapter 9, we give conditions under which the compact global attractor will be a smooth manifold taking into account the recent literature on this subject.

In the new Chapter 10 on monotonicity, we present a general class of monotone operators for which it is possible to show that the stable and unstable manifolds of hyperbolic critical elements are transversal. Applications are given to ordinary and parabolic partial differential equations as well as their time and space discretizations. This chapter also contains a presentation of the Morse decomposition of the flow on the compact global attractor for a differential delay equation with negative feedback. The proofs of all results depend in a significant way upon a discrete Lyapunov function.

Chapter 11 on the Kupka–Smale Theorem as well as the Appendix on Homotopy Index Theorem are essentially the same as in the 1984 book [87].

To assist the reader, we sometimes repeat concepts and theorems in various chapters.

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2 Invariant Sets and Attractors

In this chapter, the basic theory of invariant sets and attractors is summarized and many examples are given. Complete proofs may be found in [78] and the references cited in the text.

Suppose that X is a complete metric space with metric d and let $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^- = (-\infty, 0]$. A mapping $T : \mathbb{R}^+ \times X \to X$, $(t, x) \mapsto T(t)x$, is said to be a C^0-*semigroup* (or a *continuous semiflow*) (or a C^0-*dynamical system*) if

(i) $T(0) = I$,
(ii) $T(t + s) = T(t)T(s), t, s \in \mathbb{R}^+$,
(iii) The map $(t, x) \mapsto T(t)x$ is continuous in t, x for $(t, x) \in \mathbb{R}^+ \times X$.

For any $x \in X$, the *positive orbit* $\gamma^+(x)$ through x is defined as $\gamma^+(x) = \cup_{t\geq 0}T(t)x$. A *negative orbit* $\gamma^-(x)$ through x is the image $y(\mathbb{R}^-)$ of a continuous function $y : \mathbb{R}^- \to X$ such that, for any $t \leq s \leq 0$, $T(s - t)y(t) = y(s)$. A *complete orbit* $\gamma(x)$ through x is the union of $\gamma^+(x)$ and a negative orbit through x.

Since the range of $T(t)$ need not be the whole space, to say that there is a negative orbit through x may impose restrictions on x. Since $T(t)$ may not be one-to-one, there may be more than one negative orbit through x if one exists. We define *the negative orbit* $\Gamma^-(x)$ through x as the union of all negative orbits through x. *The complete orbit* $\Gamma(x)$ through x is $\Gamma(x) = \gamma^+(x) \cup \Gamma^-(x)$.

For any subset B of X, we let $\gamma^+(B) = \cup_{x\in B}\gamma^+(x)$, $\Gamma^-(B) = \cup_{x\in B}\Gamma^-(x)$, $\Gamma(B) = \cup_{x\in B}\Gamma(x)$ be respectively the positive orbit, negative orbit, complete orbit through B.

The limiting behavior of $T(t)$ as $t \to \infty$ is of fundamental importance. For this reason, for $x \in X$, we define $\omega(x)$, the ω-*limit set of* x or the ω-*limit set of the positive orbit through* x, as

$$\omega(x) = \cap_{\tau\geq 0}\mathrm{Cl}\gamma^+(T(\tau)x).$$

This is equivalent to saying that $y \in \omega(x)$ if and only if there is a sequence $t_k \to \infty$ as $k \to \infty$ such that $T(t_k)x \to y$ as $k \to \infty$. In the same way, for any set $B \subset X$, we define $\omega(B)$, the ω-*limit set of* B or the ω-*limit set of the positive orbit through* B, as

$$\omega(B) = \cap_{\tau\geq 0}\mathrm{Cl}\gamma^+(T(\tau)B).$$

This is the same as saying that $y \in \omega(B)$ if and only if there are sequences $t_k \to \infty$ as $k \to \infty$, $x_k \in B$, such that $T(t_k)x_k \to y$ as $k \to \infty$.

Analogously, we can define the *α-limit set of a negative orbit* $\gamma^-(x)$ *or of the negative orbit* $\Gamma^-(x)$ *of a point* x as well as the same concepts for a set $B \subset X$.

We remark that $\omega(B) \supset \cup_{x\in B}\omega(x)$, but equality may not hold. In fact, suppose that $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function for which there is a constant M such that $xf(x) < 0$ for $|x| > M$ and consider the scalar ODE $\dot{x} = f(x)$. For each $x \in \mathbb{R}$, $\omega(x)$ is an equilibrium point. If the zeros of f are simple, then, for any interval B containing at least two equilibrium points, the set $\cup_{x\in B}\omega(x)$ is disconnected, whereas $\omega(B)$ is an interval. For $f(x) = x - x^3$, $B = [-2, 2]$, we have $\cup_{x\in B}\omega(x) = \{0, \pm 1\}$, whereas $\omega(B) = [-1, 1]$.

To state a result about ω-limit sets, we need some additional notation. A set $A \subset X$ is said to be *invariant* (under the semigroup T) if $T(t)A = A$ for $t \geq 0$. We say that a set A *attracts* a set B under the semigroup T if $\lim_{t\to\infty} \mathrm{dist}_X (T(t)B, A) = 0$, where

$$\mathrm{dist}_X (B, A) = \sup_{x\in B} \mathrm{dist}_X (x, A) = \sup_{x\in B} \inf_{y\in A} \mathrm{dist}_X (x, y).$$

Lemma 2.0.1. *If $B \subset X$ is a nonempty bounded set for which there is a compact set J which attracts B, then $\omega(B)$ is nonempty, compact, invariant and attracts B. In addition, if $\omega(B) \subset B$, then*

$$\omega(B) = \cap_{t\geq 0} T(t)B.$$

In particular, if $B \subset X$ is a nonempty subset of X and there is a $t_0 > 0$ such that $\mathrm{Cl}\,\gamma^+(T(t_0)B)$ is compact, then $\omega(B)$ is nonempty, compact, invariant and $\omega(B)$ attracts B. If B is connected, then $\omega(B)$ is connected.

A compact invariant set A is said to be the *maximal compact invariant set* if every compact invariant set of T is contained in A. An invariant set A is said to be a *compact global attractor* if A is a maximal compact invariant set which attracts each bounded set of X. Notice that this implies that $\omega(B) \subset A$ for each bounded set B. It is easy to verify the following result.

Lemma 2.0.2. *If $\mathcal{A}(T)$ is compact, then $\mathcal{A}(T)$ is the maximal compact invariant set. If, for each $x \in X$, $\gamma^+(x)$ has compact closure, then $\mathcal{A}(T)$ attracts points of X. If, for any bounded set $B \subset X$, $\omega(B)$ is compact and attracts B, then $\mathcal{A}(T)$ is the compact global attractor. If $T(t)$ is one-to-one on $\mathcal{A}(T)$ for each $t \geq 0$, then T is a continuous group on $\mathcal{A}(T)$.*

To proceed further, we need some concepts of stability of an invariant set J of a continuous semigroup T. The set J is *stable* if, for any neighborhood V of J, there is a neighborhood U of J such that $T(t)U \subset V$ for all $t \geq 0$. The set J *attracts points locally* if there is a neighborhood W of J such that J attracts points of W. The set J is *asymptotically stable* if it is stable and attracts points locally. The set J is a *local attractor* or, equivalently, *uniformly asymptotically stable* if it is stable and attracts a neighborhood of J.

Lemma 2.0.3. *An invariant set J is stable if, and only if, for any neighborhood V of J, there is a neighborhood $V' \subset V$ such that $T(t)V' \subset V'$ for all $t \geq 0$. A compact invariant set J is a local attractor if and only if there is a neighborhood V of J with $T(t)V \subset V$ for all $t \geq 0$ and J attracts V.*

The following basic result on the existence of the maximal compact invariant set is due to Hale, LaSalle and Slemrod (see [83]).

Theorem 2.0.4. *If the semigroup T on X is continuous and there is a nonempty compact set K that attracts compact sets of X and $A = \cap_{t \geq 0} T(t)K$, then A is independent of K and*

(i) A is the maximal compact invariant set,
(ii) A is connected if X is connected,
(iii) A is stable and attracts compact sets of X.

It is possible to have a semigroup satisfying the conditions of Theorem 2.0.4 and yet the set $\mathcal{A}(T)$ is not a compact global attractor even though it is the maximal compact stable invariant set. As noted by Hale [80],this can be seen for linear semigroups on a Banach space X. In the statement of the result, we let $r(E\sigma(A))$ denote the radius of the essential spectrum of a linear operator A on a Banach space.

Theorem 2.0.5. *If T is a linear C^0-semigroup on a Banach space X and the origin $\{0\}$ attracts each point of X, then*

(i) $\{0\}$ is stable, attracts compact sets and is the maximal compact invariant set.
(ii) $\gamma^+(B)$ is bounded if B is bounded.
(iii) $\{0\}$ is the compact global attractor if, and only if, there is a $t_1 > 0$ such that $r(E\sigma(T(t_1))) < 1$.
(iv) If there is a t_1 such that $r(E\sigma(T(t_1))) = 1$, then the origin attracts compact sets, but is not a compact global attractor.

If X is a Banach space and T is a continuous linear semigroup for which there is a $t_1 > 0$ such that $T(t_1)$ is a completely continuous operator, then $r(E\sigma(T(t_1))) = 0$. Property (iii) of Theorem 2.0.5 implies that $\{0\}$ is the compact global attractor if it attracts each point of X.

We give two examples of interesting evolutionary equations which satisfy the conditions in (iv) of Theorem 2.0.5.

Example 2.0.6. (Neutral delay differential equation). Consider the neutral delay differential equation

$$\frac{d}{dt}[x(t) - ax(t-1)] + cx(t) = 0, \quad t \geq 0, \tag{2.1}$$

where c, a are constants. For any $\varphi \in X = C([-1,0],\mathbb{R})$, X with the sup norm, we can use this equation to define a function $x(t,\varphi)$, $t \geq -1$, with