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Reversibility in Dynamics and Group Theory

Anthony G. O'Farrell and Ian Short

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Reversibility in Dynamics and Group Theory

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Preface

Reversibility is the study of those elements of a group that are conjugate to their own inverses. Reversible maps appear naturally in classical dynamics; for instance, in the pendulum, the n-body problem, and billiards. They also arise in less obvious ways in connection with problems in geometry, complex analysis, approximation, and functional equations. When a problem has a connection to a reversible map, this opens it to attack using dynamical ideas, such as ergodic theory and the theory of flows. Reversibility has its origins in work of Birkhoff, Arnol'd, Voronin, Sevryuk, Siegel, Moser, Smale, and Devaney, among others, mainly in the context of continuous dynamical systems [8, 9, 10, 11, 31, 32, 66, 131, 211, 214, 237]. Devaney initiated the formal study of smooth reversible systems, not necessarily derived from a Hamiltonian, and there has been considerable work on such systems. The main focus has been on higher dimensions, and the systematic study of discrete reversible systems in low dimensions is more recent. We concentrate here on the discrete system theory, and on developments since the turn of the century.

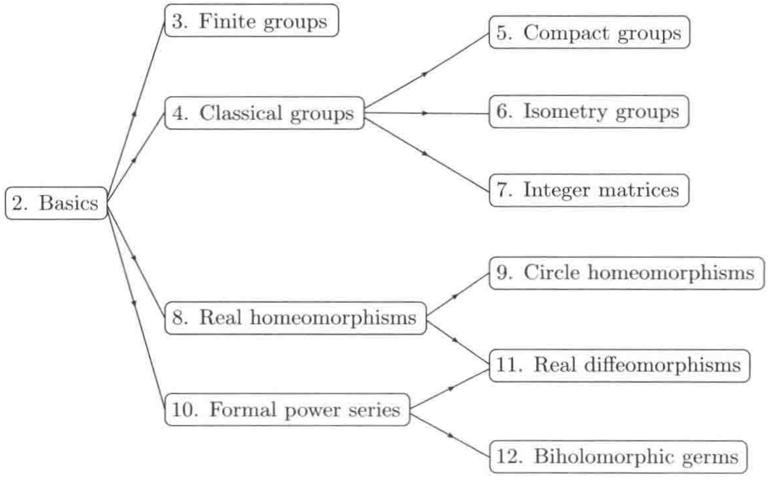
The subject relates to involutions, conjugacy problems, and automorphism groups. The reversible elements of a group are those elements that are conjugate to their own inverses, and the strongly-reversible elements are those elements that are conjugate to their own inverses by involutions. Both types of element have been studied in many contexts. For finite groups, the terms *real* and *strongly real* are used instead of *reversible* and *strongly reversible* [117, Section 9.1] because of the connections with real characters. Questions of reversibility for classical groups have been addressed in works such as [82, 150, 151, 153]. The authors of these papers use the term *bireflectional* to describe a group comprised entirely of strongly-reversible maps. There is a rich modern literature on reversibility in dynamical systems, which includes [16, 66, 159, 160, 162, 163, 164, 165, 198, 199, 205, 211]. Some authors in this field describe reversible elements as *weakly reversible*, and describe strongly-reversible elements as *reversible*.

The book opens with a brief account of the origins of the subject in the theory of reversible systems in physics, in finite group theory, and in topics where the dynamics of a reversible map prove useful in tackling problems that have no apparent dynamic connection. Then we proceed to a rapid review of general facts about the reversible elements in a group, and the reversers of these elements. The remainder of the text is a survey of (mostly) recent work on reversibility in classes of groups in which there are often attractive geometric properties. The groups we examine are finite groups, classical groups, compact groups, isometry groups, certain groups of integer matrices, and larger groups: the homeomorphism groups of the line and circle, the diffeomorphism group of the line, formal power series groups, and groups of biholomorphic germs in one variable.

The choice of topics reflects the expertise of the authors, and there are substantial results about some groups that we omit in order to keep this work within reasonable limits. These include groups of polynomial automorphisms (see for instance [15, 18, 103, 142]) and area preserving and symplectic maps (see [17, 161, 204] and, in particular, the survey by Lamb and Roberts [164]). We also neglect the long history of reversibility in ergodic theory. Arguably, the relationship between reversibility and ergodic theory began with the work of Halmos and von Neumann [127], who proved that in the group of invertible measure-preserving transformations of a Borel probability space, those transformations that are ergodic and have a discrete spectrum are strongly reversible. Halmos and von Neumann suggested that perhaps every element of this group is reversible; however, this was shown not to be so by Anzai [7]. This work was continued by Goodson, del Junco, Lema'nczyk, and Rudolph [108] who found remarkably weak conditions in the group of invertible measure-preserving transformations that ensure that the conjugating map of a reversible transformation is an involution (so the transformation is strongly reversible). For more on this topic, consult the work of Goodson [107, 108, 109, 110, 111, 112, 115], and for some recent applications of reversibility in ergodic theory, see [1, 91]. Another significant collection of groups that we omit are the finite simple groups. The finite simple groups that consist entirely of reversible (real) elements were classified by Tiep and Zalesski in [231], and it has recently been shown [22, 81, 93, 121, 122, 153, 154, 202, 233] that these finite simple groups composed entirely of reversible elements are in fact composed entirely of strongly-reversible elements.

We only scratch the surface of discrete reversible systems, and give a taste of their applications. There are several excellent accounts in books and surveys that are almost completely disjoint from ours, such as Sevryuk [211]. The survey of Lamb and Roberts [164] includes a substantial bibliography for the period up to 1998. A good deal of the work is primarily concerned with physics, such as that of Hawking, Lahiri, MacKay, Roy, Wigner, and some of Penrose. We have little or nothing to say about this, nor about the equally interesting philosophical aspects of reversibility, as reviewed and discussed, for instance, in Nickel's contribution to [83]. The subject of reversibility is massive, and we cannot include all references. Major contributions have been omitted. We include a large bibliography with sources most relevant to the material we cover, and the reader should refer to the above texts for references to other works.

The main dependencies between the chapters are described by the directed graph below. One chapter depends on another if and only if there is a directed path from the second of these chapters to the first. Chapter 2 (represented by '2. Basics' in the graph) contains a small number of definitions that are used throughout the text, but for the most part, notation is local to chapters.



Each chapter finishes first with a Notes section, which includes references and further material, and then an Open problems section. The Open problems section contains unresolved issues about reversibility from that chapter, and we hope this section will prove useful as a source of research problems, particularly for doctoral students.

In general, when we do not give the proof of a proposition, we indicate this by putting the usual QED symbol □ right after the statement. If a mature mathematician, the reader may take this as an indication that the proof is straightforward. If a student, he or she may take it as a suggestion for an exercise. In case we are quoting a substantial result from another source, without proof, we will give the reference. We write the composition of elements f and g of a group by fg. Sometimes we compose elements of a group with functions that do not belong to the group, in which case we often use the symbol $\circ$ to denote composition, for clarity. Occasionally we use the symbol $\cdot$ for multiplication, when there is a chance that multiplication may be confused with group composition.

The content of this book owes much to the advice and help of our research collaborators, especially Patrick Ahern, Nick Gill, Roman Lávička, Frédéric Le Roux, Maria Roginskaya, and Dmitri Zaitsev. We are also grateful to sometime members of the Reversible Maps Group (Mary Boyce, Mary Hanley, Ying Hou, Simon Joyce, Dennis O'Brien, Jesús San Martin, David Walsh, Richard Watson) and other participants in our seminars, who helped us to refine some of the ideas presented here. Special thanks are due to Javier Aramayona, Stefan Bechtluft-Sachs, Kurt Falk, Xianghong Gong, John Murray, Azadeh Nikou, and Claas Röver. It is a pleasure to acknowledge the support of Janice Love and Anthony Waldron, on IT matters, and of Gráinne O'Rourke for administrative backup. We are grateful to NUI, Maynooth, and also to Science Foundation Ireland, and the European Science Foundation, which provided financial support for our research, under grants RFP/05/MAT0003 and the HCAA Network, respectively. We thank the Open University and the London Mathematical Society for funding a research visit in September 2011. The first author would also like to thank Tirthankar Bhattacharya and the Indian Institute of Science, Bangalore, and Caroline Series and the Mathematical Institute, Warwick, for their hospitality while the work was in gestation. We would both like to acknowledge the help and support of the series editor, the anonymous referees, and the editorial staff at Cambridge University Press. More than anyone else, we owe an unmeasurable debt to our beloved Lise and Ellie.

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Contents

1
Origins

To motivate our study of reversibility, we describe how the concept originates in dynamical systems, finite group theory, and in a subject known as *hidden dynamics*. Full details of these topics are beyond the scope of this book, and none of the material in this chapter is needed later on.

1.1 Origins in dynamical systems

Here we discuss several examples of reversibility in the study of conservative dynamical systems.

1.1.1 The harmonic oscillator

The simple pendulum is approximately modelled by the harmonic oscillator: the system in which a particle on the real line $\mathbb{R}$ is attracted to the origin by a force directly proportional to its distance from the origin. This system also models a weight suspended from a spring, oscillating about its equilibrium position (in which case the relationship between the force and distance is given by Hooke's law). Newton's second law states that the rate of change of the momentum of a body is equal to the force applied to it. Momentum is mass times velocity, so this gives the differential equations

$$\begin{aligned} \frac{dp}{dt} &= -\kappa q, \\ \frac{dq}{dt} &= \frac{p}{m}, \end{aligned} \tag{1.1}$$

where q represents the position of the particle, p its momentum (both p and q are functions of time t), κ is the constant of proportionality between the force and the distance to the origin, and m is the particle's mass.

It follows at once that the quantity

$$H(q,p) = \frac{p^2}{2m} + \frac{\kappa q^2}{2},$$

called its *Hamiltonian* (which is, physically, the energy of the system, given by the sum of its kinetic and potential energy), has derivative zero with respect to time, and hence is constant along trajectories. It follows that the trajectories are the concentric ellipses $H(q,p) = E$, for constant $E \geqslant 0$.

Consider the map $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\tau(q,p) = (q,-p)$. Evidently, $\tau \circ \tau = \mathbb{1}$, the identity map. A simple calculation establishes the following result.

Lemma 1.1 *If* $(q(t), p(t))$ *is a solution of the differential equations* (1.1), *then so is* $\tau(q(-t), p(-t))$. □

This lemma is usually expressed as saying that τ is a *time-reversal symmetry* of the system.

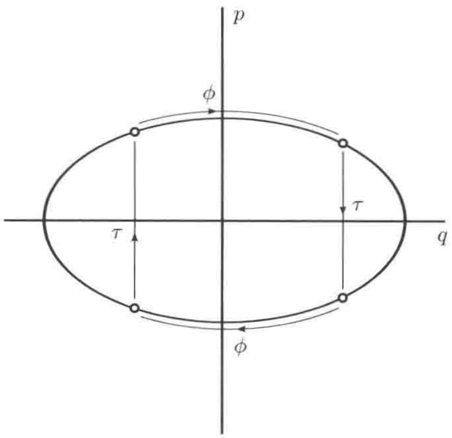

Figure 1.1 Time-reversal symmetry of the harmonic oscillator

Let $t \mapsto (q(t), p(t))$ represent the solution of (1.1) subject to the initial conditions $(q(0), p(0)) = (q_0, p_0)$, where (q_0, p_0) is some pair in $\mathbb{R}^2$. We define $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ to be the *time-one step of the system*, given by $\phi(q_0, p_0) = (q(1), p(1))$. Then

$$\phi \circ \tau \circ \phi \circ \tau = \mathbb{1},$$

or

$$\tau \circ \phi \circ \tau = \phi^{-1}, \tag{1.2}$$

the inverse map of ϕ (see Figure 1.1).

1.1.2 The n-body problem

The above behaviour is not particular to the harmonic oscillator. We can make similar observations whenever the Hamiltonian $H(q,p)$ of a dynamical system is quadratic in the momentum variable p.

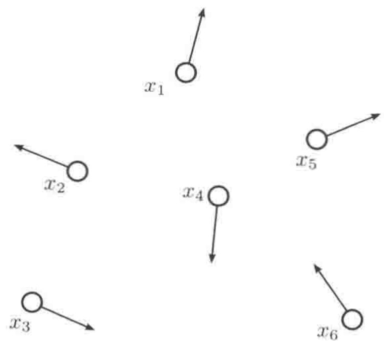

Figure 1.2 The n-body problem

Consider, for instance, the problem of n point bodies moving under their mutual gravitational attraction, illustrated in Figure 1.2. If we denote the masses by m_i and the positions by $x_i : \mathbb{R} \to \mathbb{R}^3$ ($i = 1, \ldots, n$), then in Newtonian form the equations of motion are

$$\frac{d}{dt}\left(m_i \frac{dx_i}{dt}\right) = \sum_{\substack{r=1 \\ r\neq i}}^{n} \frac{Gm_i m_r}{|x_r - x_i|^2}\left(\frac{x_r - x_i}{|x_r - x_i|}\right),$$

where G is the gravitational constant. Let $x_i = (x_{i1}, x_{i2}, x_{i3})$ for $i = 1, \ldots, n$ and, for $j = 1, 2, 3$, let

$$\mu_{3i-3+j} = m_i, \quad q_{3i-3+j} = x_{ij}, \quad p_{3i-3+j} = m_i \frac{dx_{ij}}{dt}.$$

We also define

$$K(p) = \sum_{r=1}^{3n} \frac{p_r^2}{2\mu_r}, \quad V(q) = -\frac{1}{2} \sum_{\substack{r,s=1 \\ r\neq s}}^{n} \frac{Gm_r m_s}{|x_r - x_s|},$$

$$H(q,p) = K(p) + V(q),$$

where $p = (p_1, \ldots, p_{3n})$ and $q = (q_1, \ldots, q_{3n})$. Then the equations of motion become

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k},$$
$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k},$$

for $k = 1, \ldots, 3n$. We have, as before, that $H(q, -p) = H(q, p)$, and that if $(q(t), p(t))$ is a solution, then so is $(q(-t), -p(-t))$.

This system has singularities when $n > 1$, some corresponding to collisions, and, for $n \geqslant 4$, some corresponding to other singularities [248]. Let us consider not the full phase space $\mathbb{R}^{3n} \times \mathbb{R}^{3n}$, but the subset X obtained by removing all orbits that end in a singularity, and all orbits that when run backwards end in a singularity. (By running an orbit $(q(t), p(t))$ backwards, we mean taking the orbit $(q(-t), -p(-t))$.) We remark that X is nonempty, but its structure is not fully understood to date [84].

Again, we can define $\tau(q, p) = (q, -p)$ and $\phi : X \to X$ to be the time-one step of the system, so that (1.2) holds.

1.1.3 Billiards

Consider billiards on an arbitrary smoothly-bounded, strictly-convex table without pockets. Let Γ denote the boundary. We ignore the motion in which the ball

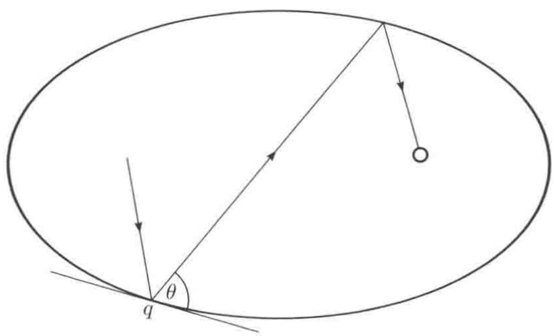

Figure 1.3 Trajectory of a billiard ball

rolls around the cushion, considering only trajectories in which it bounces to and fro. We assume that it moves in a straight line between bounces, and that at each bounce the line of incidence and the line of departure make equal angles with the normal to the boundary at the point of impact.