Asymptotic Methods in Nonlinear Wave Theory

Alan Jeffrey
Takuji Kawahara

Asymptotic Methods in Nonlinear Wave Theory

Alan Jeffrey
Department of Engineering Mathematics,
University of Newcastle upon Tyne,
England

Takuji Kawahara
Department of Physics,
Faculty of Science,
University of Kyoto,
Kyoto, Japan

Pitman Advanced Publishing Program
BOSTON · LONDON · MELBOURNE

PITMAN BOOKS LIMITED
128 Long Acre, London WC2E 9AN

PITMAN PUBLISHING INC.
1020 Plain Street, Marshfield, Massachusetts

Associated Companies
Pitman Publishing Pty Ltd, Melbourne
Pitman Publishing New Zealand Ltd, Wellington
Copp Clark Pitman, Toronto

First published 1982

AMS Subject Classifications: (main) 35B25, 35B40, 35K99
(subsidiary) 35L99

Library of Congress Cataloging in Publication Data

Jeffrey, Alan.
Asymptotic methods in nonlinear wave theory.
(Applicable mathematics)
Includes index.
1. Wave-motion, Theory of. 2. Differential equations, Nonlinear—Asymptotic theory. I. Kawahara, Takuji, 1940– . II. Title. III. Series.
QA927.J39 531'.1133 81-10569
ISBN 0-273-08509-3 AACR2

British Library Cataloging in Publication Data

Jeffrey, Alan
Asymptotic methods in nonlinear wave theory.
—(Applicable mathematics series)
1. Wave-motion, Theory of
2. Nonlinear theory
I. Title II. Kawahara, Takuji III. Series
531'.1133 QA927
ISBN 0-273-08509-3

Filmset and printed in Northern Ireland at The Universities Press (Belfast) Ltd., and bound at The Pitman Press, Bath, Avon.

Preface

In this book we have attempted to present the fundamental ideas underlying certain aspects of a number of asymptotic perturbation methods which have been developed during the last decade or so. Our objective when writing was more to identify and discuss some of the concepts which form the basis of perturbation techniques currently being used to obtain an asymptotic description of nonlinear dispersive waves, than to present a complete survey of this area.

We have intentionally offered only a formal development of the subject, even when rigour would have been possible, since we wished to keep the presentation at an heuristic level. This approach seemed to us to be more in keeping with a book which, it is hoped, will be helpful to those wishing to apply these methods to physical problems.

There are many different ways in which a work of this nature could be organized. Our plan, where possible, has been to take as the unifying theme the application of these various perturbation methods to a single equation; namely, the Boussinesq equation. This is a simple but typical and nontrivial nonlinear dispersive equation which is, in many ways, a prototype higher order nonlinear dispersive equation of the type for which the methods we discuss were developed. By examining this equation at length, and by means of different methods, all of the general ideas that need to be emphasized can be well illustrated, as can the relative merits of the methods themselves. With this discussion of the Boussinesq equation in mind, it should not be difficult for the reader to apply these same ideas to other nonlinear dispersive equations.

The structure of the book falls naturally into three distinct sections, which we have called Parts I, II and III. Each of these parts is prefaced by a brief summary of the contents. Chapter 1, which precedes Part I, is to be regarded in the nature of an elementary introduction both to wave propagation problems in general, and to issues to be developed throughout the rest of the book in particular.

In Part I itself, a number of fundamental concepts are introduced and the precise nature of a singular perturbation problem is made clear. For simplicity, this is accomplished first by using an ordinary differential equation governing a simple oscillation problem, rather than by using a partial differential equation. By means of this problem it becomes possible to introduce a number of concepts which are later shown to be directly relevant to wave problems. We demonstrate, for example, how the difficulty of nonuniformity in secular-type singular perturbation problems may be overcome. Other important issues basic to perturbation problems are also discussed.

In Part II a discussion is given of various asymptotic methods which have been applied to derive the far-field behaviour of scalar nonlinear dispersive equations, and also of nonlinear systems of equations of high order. Included in this part is an account of the reductive perturbation method. As already remarked, each different perturbation method is, so far as is possible, explained on the basis of the Boussinesq equation. The approach employed is to emphasize that many of the fundamental perturbation techniques encountered in Part I in connection with oscillation problems may be applied with only minor modification to wave problems. However, some additional ideas are required for the study of certain aspects of nonlinear dispersive waves, and these are introduced as and when they become necessary.

Part III is devoted to a number of special topics which are of sufficient importance to justify being singled out for detailed discussion. In particular, one of the main items is the method of multiple scales, together with some related ideas. The complicated topic of exact solutions is also surveyed briefly, since it now plays such an important part in the study of nonlinear dispersive wave propagation.

The bibliography, although extensive, is by no means exhaustive. Entries have been selected by the authors mainly because they seemed to be the most appropriate, or because the account contained in them was especially helpful in understanding the subject matter. Most of them are referenced in the body of the text, by author and year, at the place at which they seem most relevant. However, some have been listed at the end of sections or chapters where, being prefaced by a few general remarks, it is hoped they will provide useful source material for ideas beyond the scope of the text. A few items are only to be found in the bibliography, and these are of a general nature which offer background information, usually related to the general field of asymptotic expansions.

This book was started when one of the authors (T. K) was a Senior Visiting Fellow at the University of Newcastle upon Tyne, sponsored by the Science Research Council, London. It was completed when the other author (A. J) was on sabbatical leave at Stanford University in the USA. Thanks are due to the colleagues of both authors for helpful advice and comment. One author (A. J) would particularly like to express his gratitude to Professor J. B. Keller of Stanford University for offering him a place in his group while this work was being completed. The other author (T. K) would also like to express his thanks to Professor Tomomasa Tatsumi of Kyoto University for his continual encouragement throughout the writing of this book.

April 1981

Alan Jeffrey
Takuji Kawahara

Contents

Preface

1 Introduction to wave propagation 1
1.1 Waves and dispersion 1
1.2 Relatively undistorted waves and far fields 3
1.3 Nonlinear equations and conservation laws 6
1.4 Some important far-field equations 9
1.5 High-order systems 12
1.6 Singular perturbations of hyperbolic type 12

Part I Fundamental singular perturbation methods and nonlinear oscillations

2 Introduction to singular perturbation methods 19
2.1 Asymptotic series and perturbations 19
2.2 Secular perturbation 22
2.3 Poincaré's method 26
2.4 The method of strained coordinates 28
2.5 The method of averaging 30
2.6 The Krylov–Bogoliubov–Mitropolsky (KBM) method 33
2.7 The method of multiple scales 35
2.8 The variational method 37
2.9 Higher approximations 40
2.10 Discussion 40

Part II Various asymptotic methods useful in nonlinear wave theory

3 Asymptotic equations and perturbation methods in nonlinear wave theory 49
3.1 Fundamental concepts in wave problems 49
3.2 The formal derivation of asymptotic equations 58
3.2.1 The formal derivation of the nonlinear Schrödinger equation 59
3.2.2 The long-wave approximation 62

3.2.3 The differential approach 63
3.2.4 The Fourier approach 66
3.2.5 Reduction to the Korteweg–de Vries (KdV) equation 69
3.3 The reductive perturbation method 70
3.3.1 Nonlinear wave modulation 71
3.3.2 The long-wave approximation 73
3.3.3 General formulation 74
3.3.4 The long-wave approximation for systems 79
3.4 The Krylov–Bogoliubov–Mitropolsky (KBM) method 81
3.4.1 Wave modulation 81
3.4.2 The long-wave approximation 87
3.5 The method of multiple scales 90
3.5.1 Wave modulation 91
3.5.2 The long-wave approximation 97
3.5.3 Application to a general system 98

4 The Lagrange formalism 106
4.1 Fundamental concept 106
4.2 Wave modulation 108
4.3 Lagrangian formalism in terms of multiple scales 112
4.4 The long-wave approximation 118
4.5 The perturbation method due to Ostrovsky *et al.* 120

5 Geometrical optics—ray method 130
5.1 Outline of method 130
5.2 Wave modulation 135
5.3 The long-wave approximation 138

Part III Special topics in nonlinear wave theory

6 Two-timing and related methods 143
6.1 The two-timing procedure 143
6.1.1 The averaged Lagrangian 143
6.1.2 The two-timing technique due to Luke 145
6.1.3 Multiphase modes 148
6.2 The method of multiple scales 151
6.2.1 The derivative expansion method 152
6.2.2 The Fourier transformation 155
6.3 Applications of the derivative expansion method to wave interactions 157
6.3.1 Wave–wave interaction 161
6.3.2 Harmonic resonance 164
6.3.3 Short- and long-wave interaction 165

6.4 The multiple scale Fourier transformation 166
6.4.1 Wave modulation 168
6.4.2 The long-wave approximation 172

7 Inhomogeneous media and higher approximations 175
7.1 Wave propagation in inhomogeneous media 175
7.1.1 Nonrandom inhomogeneity 175
7.1.2 Random inhomogeneity 180
7.1.3 Effects of dissipation and instability 187
7.2 Higher approximations 190
7.2.1 The dressed soliton 191
7.2.2 The renormalization technique 194
7.3 The extension of dependent and independent variables 200
7.3.1 A secular-type problem 200
7.3.2 A singular-type problem 206

8 Methods of exact solution 211
8.1 Solutions of the Korteweg–de Vries equation 211
8.2 The inverse scattering method 213
8.3 Lax's generalization 220
8.4 The scheme due to Ablowitz *et al.* (AKNS scheme) 222
8.5 The transformation of equation method 224
8.6 Hirota's method 226
8.7 The Bäcklund transformation 230

References 237
Index 253

1 Introduction to wave propagation

1.1 Waves and dispersion

The concept of a wave is an extremely general one. It includes the cases of a clearly identifiable disturbance, either localized or not, which propagates with increasing time, a time-dependent disturbance that may or may not be repetitive in nature and which frequently has no persistent geometrical feature which can be said to propagate, and even periodic behaviour which is independent of time. Physical examples in these three categories are, respectively, the propagation of an acoustic pulse in a solid, the behaviour of a random pattern of waves on the surface of water and the undular pattern of sand bars in an estuary.

Certainly the most important single feature which characterizes a wave when time is involved, and which separates wavelike behaviour from the mere dependence of a solution on time, is that some attribute of it can be shown to propagate through space at a finite speed. In the time-dependent situations, the partial differential equations most closely associated with wave propagation are of hyperbolic type, and they may be either linear or nonlinear. However, other types of equation can also be regarded as describing wave propagation in a general sense, like certain parabolic equations which contain nonlinear terms. Their role in the study of nonlinear wave propagation is important, and knowledge of the properties of their solutions, both qualitative and quantitative, is of considerable value when applications to physical problems are involved. Such equations frequently arise as a result of the determination of the asymptotic behaviour of either a high-order equation or a complicated system of equations, and they are then often called far-field equations. Indeed, in Part II we shall discuss the systematic derivation of such equations from a system by approaches such as the reductive perturbation method and the derivative expansion technique.

Nonlinearity in waves manifests itself in a variety of ways, and in the case of waves governed by hyperbolic equations, the most striking is the evolution of discontinuous solutions from arbitrarily well behaved initial data. In the case of other types of equation, the effect of nonlinearity is usually tempered by the effects of any dispersion and dissipation that might also be present, examples of which will be given later. In general, when dispersive effects are weak, long-wavelike behaviour is possible, whereas when they are strong a highly oscillatory behaviour occurs, though the envelope of such oscillations often exhibits some of the characteristics of long waves. Apart from leading to the decay of a disturbance, dissipation also exerts a smoothing effect and, like dispersion, can combine with the effect of nonlinearity to make travelling (progressive) waves possible.

The simplest equation describing a wave is the linear equation

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0 \tag{1.1.1}$$

where $c=$ constant. For the initial data $u(x, 0)=u_0(x)$ this is easily seen to have the solution

$$u(x, t)=u_0(x-ct). \tag{1.1.2}$$

As the argument of $u_0(x-ct)$ in (1.1.2) is constant along the lines $x-ct=\xi=$ constant, this means that the initial waveform u_0 is translated at speed c without change of shape. If x is a spatial coordinate and t is the time, then c has the dimensions of speed, and a disturbance such as this which propagates at a finite speed without change of shape is called a travelling (progressive) wave. It is of fundamental importance to the understanding of many types of wave propagation. The family of lines $\xi=$ constant are themselves called the characteristic curves of equation (1.1.1) which is hyperbolic in type and, in this case, they form a family of parallel straight lines.

It is because each element $u_0(\xi)$ of the initial data is transported along the characteristic of (1.1.1) passing through the point $(\xi, 0)$ of the initial line, and in this case all of these characteristic curves are parallel straight lines, that the wave experiences no change of shape as it advances with time. This is an example of dispersionless (undistorted) wave propagation.

In order to make precise the notions of dispersion and dissipation, let us consider the simplest case of a linear operator L and the equation $L(u)=0$ for the function $u(x, t)$ in one space dimension x and time t. Then, using complex notation, any harmonic one-dimensional plane wave $\tilde{u}(x, t)$ of amplitude A and wavenumber m, moving with speed c, may be written in the form

$$\tilde{u}(x, t)=\operatorname{Re}\{A\exp[\mathrm{i}\,m(x-ct)]\}. \tag{1.1.3}$$

For what follows it is convenient to rewrite this in terms of the wavelength $\lambda=2\pi/m$, the wavenumber $k=2\pi/\lambda$ and the angular frequency $\omega=2\pi c/\lambda$, when we have

$$\tilde{u}(x, t)=\operatorname{Re}\{A\exp[\mathrm{i}(kx-\omega t)]\}. \tag{1.1.4}$$

Long waves thus correspond to small k and short waves to large k.

Then, seeking plane wave solutions of this form which satisfy $L(u)=0$, we substitute $\tilde{u}$ into this equation to arrive at a compatibility condition $D(\omega, k)=0$ to be satisfied by the wavenumber k and the angular frequency ω. This compatibility condition is known as the dispersion relation for $L(u)=0$, and it is often solved for ω in terms of k to give an alternative form of the dispersion relation

$$\omega=\omega(k). \tag{1.1.5}$$

The phase velocity V_p of the plane wave is defined as $V_p=\omega/k$, and $\text{Re}(V_p)$ is the speed of propagation of geometrical features of the wave, while the group velocity $V_g=\partial\omega/\partial k$ relates to the speed of propagation of the energy of the wave, or to its analogue (*see* Lighthill [1965a], Whitham [1974]). In general $V_p \neq V_g$, and when this occurs, with V_p depending on the wavenumber k, such waves are said to be dispersive. This name arises because waves with different wavenumbers k will then propagate at different speeds. The effect of this is that when the solution to $L(u)=0$ subject to arbitrary initial data is required at time t, the result of a Fourier superposition of such waves will be to produce a wave that changes its shape with time. This did not occur with the travelling wave solution (1.1.2) satisfying (1.1.1), since it is readily checked that for this equation $V_p=V_g=c$, so that it is dispersionless.

In conclusion, we remark that in general the dispersion relation (1.1.5) gives complex ω for real k. Thus both the phase velocity and the wave amplitude will then depend on the wavenumber (or wavelength). As a result it follows directly from (1.1.4) that, if $\text{Im}(\omega)<0$, the waves will decay exponentially with time, and this attenuation process is called dissipation. Conversely, if $\text{Im}(\omega)>0$, the waves will grow exponentially with time and this growth process is called instability. In the special case that the dispersion relation (1.1.5) is a real function of a real k, and $\partial V_p/\partial k \neq 0$, neither dissipation nor instability arise, and we call this situation pure dispersion.

Nonlinear waves are more complicated, since the frequency of a wave depends not only on the wavenumber but also on the wave amplitude, and possibly on other factors as well. To achieve a rough classification, which gives a local qualitative understanding for the behaviour of solutions, we employ a similar approach. This is accomplished by first linearizing the nonlinear equation, and then applying the above-mentioned criteria to determine whether dispersion and dissipation are present.

1.2 Relatively undistorted waves and far fields

To illustrate ideas from the previous section, and also to introduce the notion of a far field, we now consider the simple linear telegraph equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial t} + bu = 0, \tag{1.2.1}$$

where a, b and c are constants.

Seeking plane harmonic wave solutions of the form (1.1.4) we find, by direct substitution into (1.2.1), that the dispersion relation $D(\omega, k) = 0$, relating ω and k, takes the form

$$D(\omega, k) \equiv \omega^2 + \mathrm{i}\, a\omega - (b + c^2k^2) = 0. \tag{1.2.2}$$

Solving this for ω gives the dispersion relation in the alternative form

$$\omega = \tfrac{1}{2}[-\mathrm{i}\, a \pm (4c^2k^2 + 4b - a^2)^{1/2}]. \tag{1.2.3}$$

The phase velocity is thus given by the complex expression

$$V_p = \omega/k = \frac{1}{2k}[-\mathrm{i}\, a \pm (4c^2k^2 + 4b - a^2)^{1/2}], \tag{1.2.4}$$

while the group velocity is given by

$$V_g = \frac{\partial\omega}{\partial k} = \pm 2c^2k/(4c^2k^2 + 4b - a^2)^{1/2}. \tag{1.2.5}$$

Thus, since $V_p \neq V_g$, and V_p depends on the wavenumber k, the telegraph equation is dispersive. Furthermore, $\mathrm{Im}(\omega) = -\mathrm{i}\, a/2$, so that the equation is dissipative if $a > 0$ and unstable if $a < 0$, for then the solution grows without bound as $t \to +\infty$. We see from (1.2.3) that for $k < k_c$, where

$$k_c = \left(\frac{a^2 - 4b}{4c^2}\right)^{1/2}, \tag{1.2.6}$$

the phase speed V_p becomes entirely imaginary. This shows that there is no long-wave propagation for $k < k_c$, the cut-off frequency, and also that $V_g < \mathrm{Re}(V_p)$ for the waves that do propagate when $k > k_c$.

A special case arises when the condition $4b = a^2$ is satisfied, for then $\mathrm{Re}(V_p) = \pm c$, so that no dispersion occurs, though the equation is still dissipative if $a > 0$ and unstable if $a < 0$. Since in this case the shape of waves of all wavenumbers (there is now no cut-off) is unchanged as they propagate, and their amplitudes all decay or grow at the same rate, such waves are said to be relatively undistorted.

The structure of harmonic plane waves (1.1.4) for the telegraph equation is as follows, and it illustrates quite clearly both the general case and the special case in which $4b = a^2$:

$$\tilde{u}(x, t) = \mathrm{Re}\left\{A \exp(-\tfrac{1}{2}at) \exp\left[\mathrm{i}\, k\left(x \pm \frac{t}{2k}(4c^2k^2 + 4b - a^2)^{1/2}\right)\right]\right\}. \tag{1.2.7}$$

The special case of the telegraph equation also enables us to illustrate some asymptotic features of equations and, in the process, to introduce the notion of a far field. If $4b = a^2$, then (1.2.1) may be written in either the form

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}+\tfrac{1}{2}a\right)\left(\frac{\partial u}{\partial t}-c\frac{\partial u}{\partial x}+\tfrac{1}{2}au\right)=0, \qquad \textbf{(1.2.8)}$$

or the form

$$\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}+\tfrac{1}{2}a\right)\left(\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}+\tfrac{1}{2}au\right)=0. \qquad \textbf{(1.2.9)}$$

Now suppose that $u^{(\pm)}$ are functions such that

$$\frac{\partial u^{(+)}}{\partial t}+c\frac{\partial u^{(+)}}{\partial x}+\tfrac{1}{2}au^{(+)}=0, \qquad \textbf{(1.2.10)}$$

and

$$\frac{\partial u^{(-)}}{\partial t}-c\frac{\partial u^{(-)}}{\partial x}+\tfrac{1}{2}au^{(-)}=0. \qquad \textbf{(1.2.11)}$$

Then (1.2.8) and (1.2.9) are automatically satisfied by the functions $u^{(\pm)}$. This shows that in addition to the $u^{(\pm)}$ being solutions of the appropriate first-order equation, they are also solutions of the telegraph equation. They are important, though degenerate, solutions of the telegraph equation. This is because whereas wave propagation in the telegraph equation is bi-directional, since it is second order in time, wave propagation characterized by the first-order equations (1.2.10) and (1.2.11) is only uni-directional. The solution $u^{(+)}$ propagates to the right and $u^{(-)}$ to the left, and they will continue to do so indefinitely. They are relatively undistorted travelling waves.

This situation should be expected, because solutions to the first-order equations (1.2.10) and (1.2.11) cannot satisfy all the initial conditions appropriate to the second-order equation (1.2.1). The reduction of order of an equation, coupled with the associated loss of the ability of the degenerate solution to satisfy all the original initial conditions, is typical of the asymptotic behaviour of an equation. In this case the solutions $u^{(\pm)}$ are easy to interpret. If the initial data for the telegraph equation are localized into a finite interval on the initial line (has compact support) then, after a suitable time interval during which waves moving to the left and right interact, the initial disturbance will have resolved itself into two distinct relatively undistorted disturbances moving in opposite directions with speed c. Thus after interaction, and suitably far to the right and left of the initial localized disturbance, the solution takes on a simplified form which we shall call the far field of the initial value problem for the telegraph equation. The equations of reduced order (1.2.10) and (1.2.11) governing the behaviour of the far fields $u^{(\pm)}$ are often called the far-field equations for (1.2.1) from (1.2.8).

In this case the far fields exist because, with a, b and c constant and $4b = a^2$, the telegraph equation may be factorized and there is commutability of the factors, leading directly to (1.2.9).

The relatively undistorted nature of these far fields may also be deduced from (1.2.10) and (1.2.11). If, for example, we consider (1.2.10), then along the characteristic curves

$$C^{(+)}: \qquad x - ct = \xi, \qquad \xi = \text{constant}, \tag{1.2.12}$$

equation (1.2.10) may be written

$$\frac{\mathrm{d}u^{(+)}}{\mathrm{d}t} + \tfrac{1}{2}au^{(+)} = 0. \tag{1.2.13}$$

Hence, integration shows that

$$u^{(+)}(t) = u^{(+)}(0)\mathrm{e}^{-at/2} \qquad \text{with } x - ct = \xi, \tag{1.2.14}$$

from which it is at once apparent that the exponential change of amplitude along all $C^{(+)}$ characteristics is the same and depends only on the time. A corresponding argument may be applied to (1.2.11) along the $C^{(-)}$ characteristics $x + ct = \eta$ where $\eta = \text{constant}$, when the identical conclusion will be reached.

The concept of the asymptotic behaviour of a nonlinear equation and of a far field may also be developed, though by quite different techniques. However, the failure of the asymptotic solution to satisfy all the original initial conditions, and the reduction of order of the governing equation itself, will still persist. We shall turn to the systematic derivation of the asymptotic behaviour of both high-order nonlinear equations and systems after the review of basic ideas in Part I.

1.3 Nonlinear equations and conservation laws

The change of shape of a wave is not necessarily only the result of dispersion, as it is an inherent consequence of nonlinearity. This is seen most simply by considering the scalar equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(f(u)) = 0, \tag{1.3.1}$$

subject to the initial data

$$u(x, 0) = u_0(x). \tag{1.3.2}$$

Performing the indicated differentiation of (1.3.1), and recognizing that

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + f'(u)\frac{\partial}{\partial x} \tag{1.3.3}$$

is a directional derivative along the characteristic curves C given by integrating

$$C: \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = f'(u), \tag{1.3.4}$$

enables (1.3.1) to be written

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \qquad \text{along the characteristics } C. \tag{1.3.5}$$

Thus u = constant, along each characteristic C, though in general it will be a different constant along each different characteristic. This result, when combined with (1.3.4), shows that the characteristics C are nonparallel straight lines provided $f'(u) \neq$ constant, which is the condition which ensures the nonlinearity of (1.3.1). As characteristics transport along them constant values of the solution u, and they are nonparallel straight lines, it follows that different elements of the initial wave profile will propagate at different speeds, so that the wave shape will change with time. It is for this very reason that nonlinear hyperbolic equations which are not dispersive in the sense already defined cannot have travelling wave solutions, as there is no permanency of shape.

Nonuniqueness of the solution results when for $t > 0$ the characteristics intersect. This is because characteristics with different slopes transport different constant values of the solution u, so that intersection implies nonuniqueness. The envelope of the characteristics in the (x, t)-plane for $t > 0$ marks the boundary of the region above the initial line in which a differentiable solution to (1.3.1) exists.

An extension of the solution beyond the time at which such nonuniqueness occurs becomes possible if the equation involved is a conservation law. This is accomplished by introducing a solution which is discontinuous across a line in the (x, t)-plane, when the conservation law makes it possible to relate the solution on one side of the discontinuity line to the solution on the other side. A conservation law for the quantity u expresses the rate of change of u in an arbitrary volume V in terms of the negative flux of $f(u)$ across the boundary ∂V of V, by means of the result

$$\frac{\partial}{\partial t}\int_V u\,\mathrm{d}V = -\int_{\partial V} f(u)\,\mathrm{d}S, \tag{1.3.6}$$

where $\mathrm{d}V$ and $\mathrm{d}S$ are volume and surface elements, respectively. For suitably differentiable u and $f(u)$, it follows from Gauss's theorem and the arbitrariness of V that

$$\frac{\partial u}{\partial t}+\operatorname{div} f(u)=0. \tag{1.3.7}$$

We see from this that (1.3.1) is, in fact, a one-dimensional conservation law.

The jumps $[\![u]\!]$ and $[\![f(u)]\!]$ across a discontinuity line in the (x, t)-plane can be shown to satisfy the so-called generalized Rankine–Hugoniot jump condition

$$[\![u]\!]=\tilde{\lambda}[\![f(u)]\!], \tag{1.3.8}$$

in which $\tilde{\lambda}$ is the speed of propagation of the discontinuity. Such a jump solution is usually called a shock wave. These same ideas extend to quasilinear hyperbolic systems of conservation type which exhibit the same form of behaviour. It is appropriate to remark here that these arguments show that it is not the change of shape of a solution representing a wave which is characteristic of quasilinear hyperbolic wave propagation, since linear dispersive waves share the same property, but rather the change of nature of the solution itself when it evolves to the point at which a discontinuity arises.

It will suffice here merely to outline the determination of the time at which the nonuniqueness of the solution to (1.3.1) subject to (1.3.2) first arises and a shock first forms.

The straight line characteristic C through the point $(\xi, 0)$ on the initial line is easily seen to have the equation

$$x=\xi+tf'(u_0(\xi)), \tag{1.3.9}$$

while along it

$$u=u_0(\xi). \tag{1.3.10}$$

An elementary calculation then shows that, when expressed in parametric form, the envelope of the characteristics with ξ as a parameter is given by

$$x=\xi+tf'(u_0(\xi)), \qquad t=\frac{-1}{f''(u_0(\xi))u_0'(\xi)}. \tag{1.3.11}$$

The following may then be deduced immediately from the expression for t in (1.3.11).

Theorem *Let $u(x, t)$ satisfy the conservation law*

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(f(u))=0,$$

subject to $u(x, 0)=u_0(x)$, with $|u_0|\leq M$, $|u_0'|\leq u^{(1)}$ and $f^{(2)}=\max|f''(\xi)|$ for