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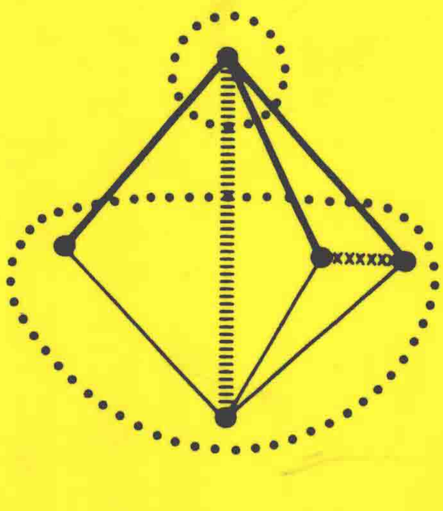
Algorithmic Graph Theory and Perfect Graphs

Second Edition



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Algorithmic Graph Theory and Perfect Graphs

Second Edition

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♾ The paper used in this publication meets the requirements of ANSI/NISO Z39.48-1992 (Permanence of Paper).
Printed in Hungary.

Dedicated to my parents

לכבוד הורי היקרים
אברם בן יצחק גולומביק
חיענע בת מנדיל הכהן

Foreword 2004: the *Annals* edition

The publication of this new edition of *Algorithmic Graph Theory and Perfect Graphs* marks twenty three years since its first appearance. My original motivation for writing the book was to collect and unify the topic to act as a spring board for researchers, and especially graduate students, to pursue new directions of investigation. The ensuing years have been an amazingly fruitful period of research in this area. To my great satisfaction, the number of relevant journal articles in the literature has grown tenfold. I can hardly express my admiration to all these authors for creating a success story for algorithmic graph theory far beyond my own imagination.

The world of perfect graphs has grown to include over 200 special graph classes. The Venn diagrams that I used to show some of the inclusions between classes in the First Generation, for example Figure 9.9 (on page 212), have yielded to Hasse diagrams for the Second Generation, like the one from Golumbic and Trenk [2004] reprinted in Figure 13.3 at the end of this edition.

Perhaps the most important new development in the theory of perfect graphs is the recent proof of the Strong Perfect Graph Conjecture by Chudnovsky, Robertson, Seymour and Thomas, announced in May 2002. News of this was immediately passed on to Claude Berge, who sadly passed away on June 30, 2002.

On the algorithmic side, many of the problems which were open in 1980 have subsequently been settled, and algorithms on new classes of perfect graphs have been studied. For example, tolerance graphs generalize both interval graphs and permutation graphs, and coloring tolerance graphs in polynomial time is important in solving scheduling problems where a measure of flexibility or tolerance is allowed for sharing or relinquishing resources when total exclusivity prevents a solution.

At the end of this new edition, I have added a short chapter called

Epilogue 2004 in which I survey a few of my favorite results and research directions from the Second Generation. Its intension is to whet the appetite.

Six books have appeared recently which cover advanced research in this area. They have thankfully relieved me from a pressing need to write my own encyclopedia sequel. They are the following, and are a must for any graph theory library.

- A. Brandstädt, V.B. Le and J.P. Spinrad, "*Graph Classes: A Survey*", SIAM, Philadelphia [1999], is an extensive and invaluable compendium of the current status of complexity and mathematical results on hundreds on families of graphs. It is comprehensive with respect to definitions and theorems, citing over 1100 references.
- P.C. Fishburn, "*Interval Orders and Interval Graphs: A Study of Partially Ordered Sets*", John Wiley & Sons, New York [1985], gives a comprehensive look at the research on this class of ordered sets.
- M.C. Golumbic and A.N. Trenk, "*Tolerance Graphs*", Cambridge University Press [2004], is the youngest addition to the perfect graph bookshelf. It contains the first thorough study of tolerance graphs and tolerance orders, and includes proofs of the major results which have not appeared before in books.
- N.V.R. Mahadev and U.N. Peled, "*Threshold Graphs and Related Topics*", North-Holland [1995], is a thorough and extensive treatment of all research done in the past years on threshold graphs (chapter 10 of my book), threshold dimension and orders, and a dozen new concepts which have emerged.
- T.A. McKee and F.R. McMorris, "*Topics in Intersection Graph Theory*", SIAM, Philadelphia [1999], is a focused monograph on structural properties, presenting definitions, major theorems with proofs and many applications.
- W.T. Trotter, "*Combinatorics and Partially Ordered Sets*", Johns Hopkins, Baltimore [1992], is the book to which I referred at the bottom of page 136. It covers new directions of investigation and goes far beyond just dimension problems on ordered sets.

Algorithmic Graph Theory and Perfect Graphs has now become the classic introduction to the field. It continues to convey the message that intersection graph models are a necessary and important tool for solving real-world problems. Solutions to the algorithmic problems on these special graph classes are continually integrated into systems for a large variety of application areas, from VLSI circuit design to scheduling, from resource allocation to physical mapping of DNA, from temporal reasoning in artificial intelligence to pavement deterioration analysis. On the mathematical side, perfect graph classes have provided rich soil for deep theoretical results. In short, it remains a stepping stone from which the reader may embark on one of many fascinating research trails.

Martin Charles Golumbic
Haifa, Israel

Foreword

Research in graph theory and its applications has increased considerably in recent years. Typically, the elaboration of new theoretical structures has motivated a search for new algorithms compatible with those structures. Rather than the arduous and systematic study of every new concept definable with a graph, the main task for the mathematician is to eliminate the often arbitrary and cumbersome definitions, keeping only the "deep" mathematical problems.

Of course, the deep problems may well be elusive; indeed, there have been many definitions (from Dieudonné, among others) of what a deep problem is. In graph theory, it should relate to a variety of other combinatorial structures and must therefore be connected with many difficult practical problems. Among these will be problems that classical algebra is not able to solve completely or that the computer scientist would not attack by himself.

This book, by Martin Golumbic, is intended as an introduction to graph theory through just these practical problems, nearly all of them related to the structure of permutation graphs, interval graphs, circle graphs, threshold graphs, perfect graphs, and others.

The reader will not find motivations drawn from number theory, as is usual for most of the extremal graph problems, or from such refinements of old riddles as the four-color problem and the Hamiltonian tour. Instead, Golumbic has selected practical problems that occur in operations research, scheduling, econometrics, and even genetics or ecology.

The author's point of view has also enjoyed increasing favor in the area of complexity analysis. Each time a new structure appears, the author immediately devotes some effort to a description of efficient algorithms, if any are known to exist, and to a determination of whether a proposed algorithm is able to solve the problem within a reasonable amount of time.

Certainly a wealth of literature on graph theory has developed by now. Yet it is clear that this book brings a new point of view and deserves a special place in the literature.

CLAUDE BERGE

Preface

The notion of a "perfect" graph was introduced by Claude Berge at the birth of the 1960s. Since that time many classes of graphs, interesting in their own right, have been shown to be perfect. Research, in the meantime, has proceeded along two lines. The first line of investigation has included the proof of the perfect graph theorem (Theorem 3.3), attempts at proving the strong perfect graph conjecture, studies of critically imperfect graphs, and other aspects of perfect graphs. The second line of approach has been to discover mathematical and algorithmic properties of special classes of perfect graphs: comparability graphs, triangulated graphs, and interval graphs, to name just a few. Many of these graphs arise quite naturally in real-world applications. For example, uses include optimization of computer storage, analysis of genetic structure, synchronization of parallel processes, and certain scheduling problems.

Recently it appeared to me that the time was ripe to assemble and organize the many results on perfect graphs that are scattered throughout the literature, some of which are difficult to locate. A serious attempt has been made to coordinate the mélange of some 200 papers referenced here in a manner that would make the subject more accessible to those interested in algorithmic and algebraic graph theory. I have tried to include most of the important results that are currently known. In addition, a few new results and new proofs of old results appear throughout the text. In particular, Chapter 9, on superperfect graphs, contains results due to Alan J. Hoffman, Ellis Johnson, Larry J. Stockmeyer, and myself that are appearing in print for the first time.

The emphasis of any book naturally reflects the bias of the author. As a mathematician and computer scientist, I am doubly biased. First, I have tried to present a rigorous and coherent theory. Proofs are constructive and are streamlined as much as possible. The notation has been chosen to facilitate these matters. Second, I have directed much attention to the algorithmic aspects of every problem.

Algorithms are expressed in a manner that will make their adaptation to a particular programming language relatively easy. The complexity of every algorithm is analyzed so that some measure of its efficiency can be determined.

These two approaches enhance one another very well. By exploiting the mathematical properties satisfied a priori by a structure, one is often able to reduce the time or space complexity required to solve a problem. Conversely, the algorithmic approach often leads to startling theoretical results. To illustrate this point, consider the fact that certain NP-complete problems become tractable when restricted to certain classes of perfect graphs, whereas the algorithm for recognizing comparability graphs gives rise to a matroid associated with the graph.

A glance at the table of contents will provide a rough outline of the topics to be discussed. The first two chapters are introductory in the sense that they provide the foundations, respectively, of the graph theoretic notions and the algorithmic design and analysis techniques that will be used in the remaining chapters. The reader may wish to read these two chapters quickly and refer to them as needed. The chapters are structured in such a way that the book will be suitable as a textbook in a course on algorithmic combinatorics, graph theory, or perfect graphs. In addition, the book will be very useful for applied mathematicians and computer scientists at the research level. Many applications of the theoretical and computational aspects of the subject are described throughout the text. At the end of each chapter there are numerous exercises to test the reader's understanding and to introduce further results. An extensive bibliography follows each chapter, and, when possible, the *Mathematical Reviews* number is included for further reference.

The topics covered in this book have been chosen to fill a vacuum in the literature, and their interrelation and importance will become evident as early as Section 1.3. Since the intersection of this volume with the traditional material covered by most graph theory books has been designed to be small, it is highly recommended that the serious student augment his studies with one of these excellent textbooks. A one-year course with two concurrent texts is suggested.

MARTIN CHARLES GOLUMBIC

Acknowledgments

I would like to express my gratitude to the many friends and colleagues who have assisted me in this project. Special thanks are due to Claude Berge for the kind words that introduce this volume. I am happy to acknowledge the help received from Mark Buckingham, particularly in Chapters 3 and 11. He is the coauthor of Sections 3.3–3.5. The suggestions and critical comments of my "trio" of students, Clyde Kruskal, Larry Rudolph, and Elia Weixelbaum, led to numerous improvements in the exposition. Over the past three years I have been fortunate to receive support from the Courant Institute of Mathematical Sciences, the National Science Foundation, the Weizmann Institute of Science, and l'Université de Paris VI.

I would also like to express my appreciation to Alan J. Hoffman for many interesting discussions and for his help with the material in Chapter 9. My thanks go to Uri Peled, Fred S. Roberts, Allan Gottlieb, W. T. Trotter, Peter L. Hammer, and Lászlò Lovász for their comments, as well as to Lisa Sabbia Walsh, Daniel Gruen, and Joseph Miller for their assistance. I am also indebted to my teacher, Samuel Eilenberg, for the guidance, insight, and kindness shown me during my days at Columbia University.

But the greatest and most crucial help has come from my wife Lynn. Although not a mathematician, she managed to unconfound much of this mathematician's gibberish. She also "axed" some of my worst (best) jokes, much to my dismay. More importantly, she has been the rock on which I have always relied for encouragement and inspiration, during our travels and at home, in the course of the research and writing of this book. As it is written in Proverbs:

פיה פתחה בחכמה, ותורת־חסד על־לשונה.
רבות בנות עשו חיל, ואת עלית על־כלנה.

List of Symbols

Page	*Symbol*	*Meaning*
1	$\forall x$	For *all* x.
1	$\exists\, y$	There *exists* a y.
1	$x \epsilon X$	x is a *member* of X.
1	$A \subseteq X$	A is a *subset* of X.
1	$B \subset X$	B is a *proper subset* of X.
1	$\lvert X \rvert$	The *cardinality* of a set X.
1	$A \cap B$	The *intersection* of A and B.
1	$A \cup B$	The *union* of A and B.
2	$A + B$	The *union* of *disjoint* sets A and B.
2	$\varnothing$	The *empty set*.
2	$\mathscr{P}(X)$	The *power set* of X.
4	$V \times W$	The *Cartesian product* of sets V and W.
19	$S \between T$	Sets S and T *overlap*; $S \cap T \neq \varnothing$, $S \not\subseteq T$, and $T \not\subseteq S$.
3	$G = (V, E)$	The *graph* G with vertex set V and edge set E.
8	$G = (X_1, X_2, E)$	The *bipartite graph* G with vertex set $X_1 + X_2$ where each X_i is stable.
5	(V_S, S)	The subgraph *spanned* by a subset S of edges.
6	$G_A = (A, E_A)$	The subgraph *induced* by a subset A of vertices.
3	$\mathrm{Adj}(v)$	The *adjacency* set of vertex v.
6	$\mathrm{Adj}_A(v)$	The adjacency set restricted to A; $\mathrm{Adj}_A(v) = \mathrm{Adj}(v) \cap A$.
3	$N(v)$	The *neighborhood* of vertex V; $N(v) = \{v\} + \mathrm{Adj}(v)$.
7	$d^{+}(v)$	The *out-degree* of vertex v.
7	$d^{-}(v)$	The *in-degree* of vertex v.
7	$d(v)$	The *degree of vertex* v in an undirected graph.
4	E^{-1}	The *reversal* of a set E of edges.
4	$\hat{E}$	The *symmetric closure* of a set E of edges.
4	$\widehat{ab}$	The *undirected edge* $\{ab\} \cup \{ba\}$.
7	$\lVert E \rVert$	In an undirected graph $G = (V, E)$ we define $\lVert E \rVert = \frac{1}{2} \lvert E \rvert$.
4	$\bar{G}$	The *complement* of an undirected graph G.
4	$G \cong G'$	Graphs G and G' are *isomorphic*.
6	$\omega(G)$	The *clique number* of G.
6	$k(G)$	The *clique cover number* of G.

6	$\alpha(G)$	The *stability number* of G.
7	$\chi(G)$	The *chromatic number* of G.
113	$t(G)$	The number of *transitive orientations* of G.
126	$r(G)$	The *rank* of the Γ^*-matroid of G.
220	$\theta(G)$	The *threshold dimension* of G.
203	$\chi(G;w)$	The *interval chromatic number* of a weighted graph $(G;w)$.
206	$\omega(G;w)$	The maximum *weighted clique number* of $(G;w)$.
9	K_n	The *complete graph* on n vertices.
9	C_n	The *chordless cycle* on n vertices.
9	P_n	The *chordless path graph* on n vertices.
9	$K_{m,n}$	The *complete bipartite graph* on $m+n$ vertices partitioned into an m-stable set and an n-stable set.
9	$K_{1,n}$	The *star graph* on $n+1$ vertices.
9	mK_n	m disjoint copies of K_n.
47	$G_1 \times G_2$	The *Cartesian product* of graphs G_1 and G_2.
77	$G \cdot H$	The *normal product* of graphs G and H.
109	$H_0[H_1, \ldots, H_n]$	The *composition* of graphs.
95	$\mathscr{G}$	The class of undirected graphs satisfying the property that every odd cycle of length greater than or equal to 5 has at least two chords.
105	Γ	The *forcing* relation on edges.
106	Γ^*	The reflexive, transitive closure of Γ.
106	$\mathscr{I}(G)$	The collection of *implication classes* of G.
106	$\hat{\mathscr{I}}(G)$	The collection of *color classes* of G.
135	$\mathscr{L}(P)$	The collection of *linear extensions* of a partial order P.
135	$\dim(P)$	The *dimension* of a partial order P.
157	$G[\pi]$	The *permutation graph* of π.
235	$H[\pi]$	The *stack sorting graph* of π.
157	π^{-1}	The *inverse* of the permutation π.
158	π^ρ	The *reversal* of the permutation π.
228	$\sqcup\!\sqcup$	The *shuffle product*.
236	$\mathscr{H}$	The class of *stack sorting graphs*.
23	$O(f(m))$	Computational complexity *on the order of* $f(m)$.
26	**P**	The class of *deterministic polynomial-time* problems.
27	**NP**	The class of *nondeterministic polynomial-time* problems.
27	$\Pi_1 \preccurlyeq \Pi_2$	Problem Π_1 is *polynomially transformable* to problem Π_2.
32	Λ	The *null* or *undefined* symbol in an algorithm.
176	$T \equiv T'$	The *PQ*-trees T and T' are *equivalent*.
177	$\Pi(\mathscr{I})$	The collection of all permutations π of X such that the members of each subset $I \in \mathscr{I}$ occur consecutively in π where $\mathscr{I} \subseteq \mathscr{P}(X)$.
53	$G \circ \mathbf{h}$	The graph G *multiplied* by the vector **h**.
62	$\mathbb{R}^n$	The n-dimensional vector space over the *real numbers*.
62	$P(\mathbf{A})$	The *polyhedron* of matrix **A**.
62	$P_I(\mathbf{A})$	The *integral polyhedron* of matrix **A**.
59	$\mathbf{1}$	The vector of all ones.
62	$\mathbf{0}$	The vector of all zeros.
60	$\mathbf{J}$	The matrix of all ones.
60	$\mathbf{I}$	The identity matrix.
256	$G(\mathbf{M})$	The graph of matrix **M**.
256	$B(\mathbf{M})$	The bipartite graph of matrix **M**.

Corrections and Errata to: *Algorithmic Graph Theory and Perfect Graphs*, the original 1980 edition

We apologize to Prof. George Lueker for misspelling his family name throughout the text. Hence all occurrences "Leuker" should be "Lueker".

Page 18: The graph in Figure 1.17 is a circular-arc graph.

Page 48: Exercise 21 is false.

Page 49: Garey and Johnson [1978]: add "MR80g:68056"

Page 78: Bland, et al. [1979]: add "MR80g:05034"
Chvátal, et al. [1979]: add "MR81b:05044"
de Werra [1978]: add "MR81a:05052"
Greenwell [1978]: add "MR80d:05044"

Page 79: Olaru [1977]: add "MR58#5411"

Page 80: Parthasarathy and Ravindra [1979]: add "MR80m:05045"
Pretzel [1979]: add "MR80d:06003"
Tucker [1979]: add "MR81c:05041"
Wagon [1978]: add "MR80i:05078"

Page 85: Figure 4.3: The edge (b, e) is missing.

Page 102: Exercise 24: The claim in the first sentence is false. For example, it can use as many as 7 colors on the graph G_1, in Figure 4.1. A different technique can be used to obtain a linear time coloring algorithm for triangulated graphs, which is due to Martin Farber.

Line 21: change "$Adj(w)$" to "$Adj(u)$"

Gavril [1978]: add "MR81g:05094"

Page 104: Wagon [1978]: add "MR80i:05078"

Page 138: The second footnote can be updated since M. Yannakakis has now proved that the complexity of determining if a poset has dimension 3 is NP-complete.

Page 145: Pretzel [1979]: add "MR80d:06003"

Page 146: Gysin [1977]: add "MR58#5393"

Page 147: Rabinovitch [1978b]: add "MR58#5424"

Page 156: Burkard and Hammer [1977]: change to the following:
[1980] A note on Hamiltonian split graphs, *J. Combin. Theory B* **28**, 245–248. MR81e:05095.

A necessary condition for the existence of a Hamiltonian cycle in split graphs is proved.

Erdos and Gallai [1960]: change "272" to "274"

Foldes and Hammer [1978]: add "MR80c:05111"

Hammer, Ibaraki, and Simeone [1978]: change to the following:
[1978] Degree sequences of threshold graphs, *Proc. 9th Southeastern Conf. on Combinatorics, Graph Theory and Computing, Congressus Numeratium* **21**, Utilitas Math., Winnipeg, Man., 329–355. MR80j:05088.

Page 163: There should be edges between 3–4 and 6–7 (corrected in this edition).

Page 179: Figure 8.7: The second tree on the right should have its rightmost leaf "F" rather than "E". The leaves should read from left to right as follows: B C E A D F

Page 190: line 6: change "will appear in Tucker [1979]" to: "appears in Tucker [1980]"

Page 197: line 26: change "Griggs and West [1979]" to: "Griggs and West [1980]"
Abbott and Katchalski [1979]: add "MR80b:05038"

Page 198: Booth and Lueker [1976]: add "MR55#6932"

Page 199: Griggs [1979]: add "MR81h:05083b"
Griggs and West: change to the following:
[1980] Extremal values of the interval number of a graph, *SIAM J. Algebraic Discrete Methods* 1, 1–7. MR81h:05083a.

Page 201: Roberts [1979a]: add "MR81e:05120"
Roberts [1979b]: add "MR81e:05071"
Trotter and Harary [1979]: add "MR81c:05055"

Page 202: Tucker [1979]: change to the following:
[1980] An efficient test for circular-arc graphs, *SIAM J. Comput.* 9, 1–24. MR81a:68074.

Page 203: line 17: add the following:
Vertices x of weight $w(x) = 0$ are the mapped into the empty interval.

Page 206: $\omega(T; w)$ should be $\omega(G; w)$

Page 212: Figure 9.9:

(1) The nonsuperperfect, interval graph with the chordless 5-cycle should have two chords connecting the top two vertices to the bottom vertex. It will then be the same as the "bull's head" graph on page 16, (corrected in this edition).

(2) The noncomparability, nontriangulated comparability graph on 7 vertices has too many edges. The two vertical edges should be removed, (corrected in this edition).

(3) The nonsuperperfect, interval graph which has 5 triangles, is, in fact, superperfect; it should be moved into the superperfect, noncompatability, interval area of the figure. See also Section 13.9 of the Epilogue to this edition.

Page 234: Golumbic [1978a]: add "MR81e:68080"
Hammer, Ibaraki, and Simeone [1978]: change to the following:
[1978] Degree sequences of threshold graphs, *Proc. 9th Southeastern Conf. on Combinatorics, Graph Theory and Computing, Congressus Numeratium* **21**, Utilitas Math., Winnipeg, Man., 329–355. MR80j:05088.

Page 253: Gavril [1973]: change "minimum independent" to "maximum independent"

Page 267: Golumbic [1979]: add "MR81c:05077"
Golumbic and Goss [1978]: add "MR80d:05037"
Ohtsuki, Cheung, and Fujisawa [1976]: add "MR58#5379"

Page 280: Lueker, G. S.: change "25" to "24"
Put name into alphabetical order.