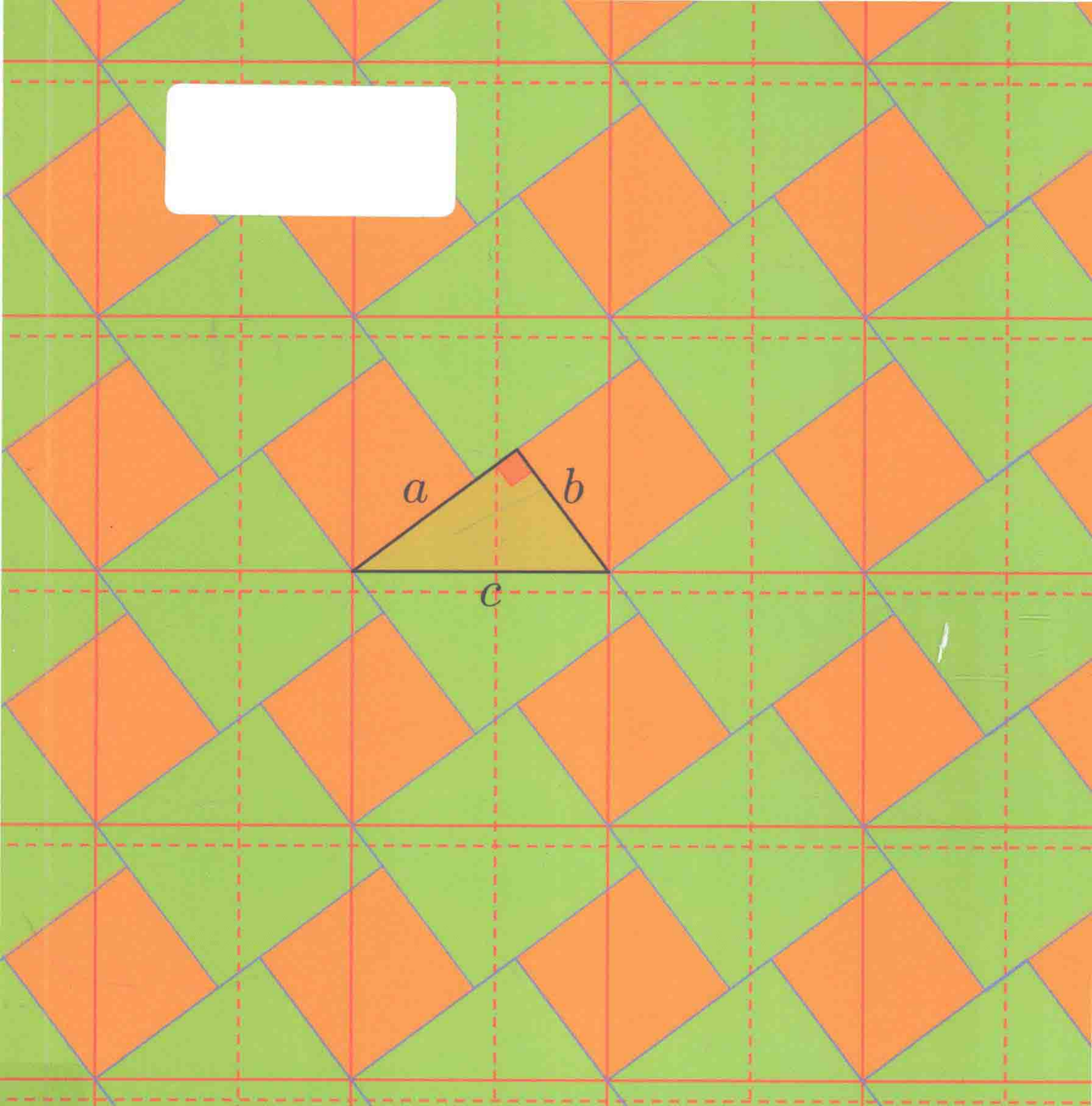
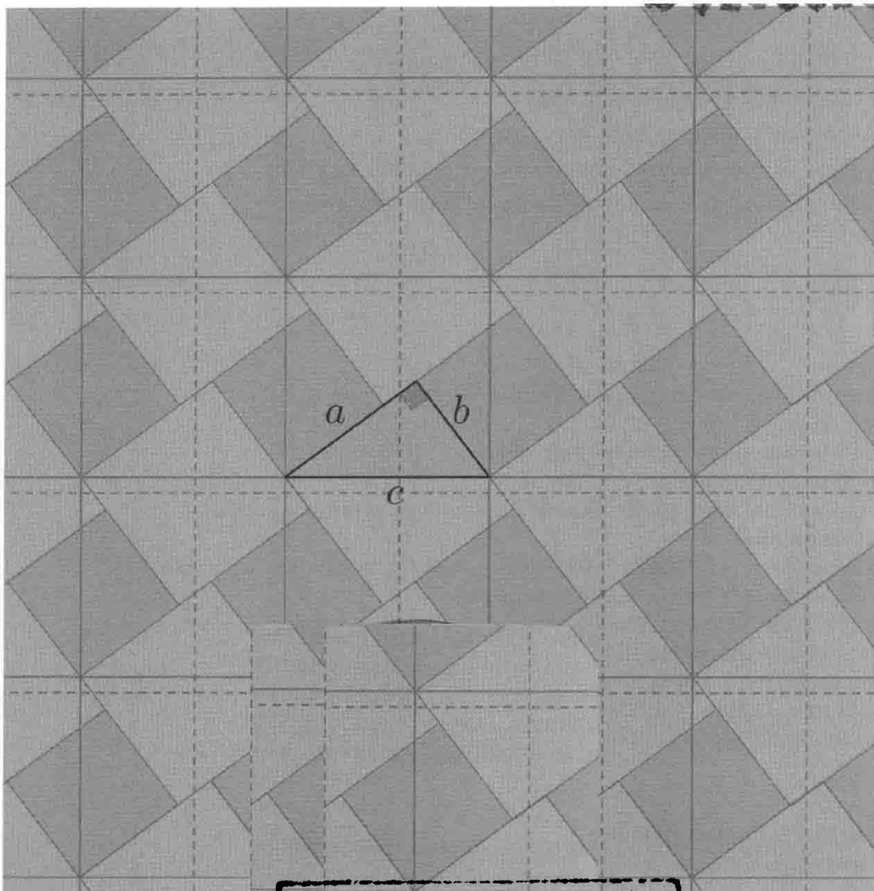


PLAIN PLANE GEOMETRY

Amol Sasane

World Scientific



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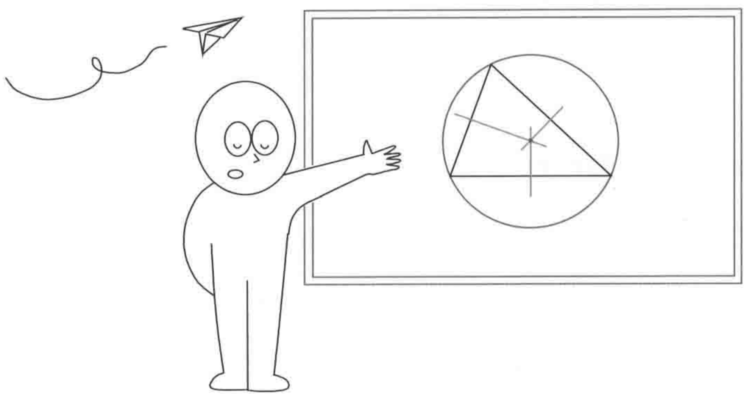
PLAIN PLANE GEOMETRY

To Arun

Preface



Why this book?

Because I want to share the feeling that planar geometry can be fun. Although as high school students, my generation were taught Euclidean plane geometry, it is common to encounter a modern-day high school mathematics textbook which is largely devoid of any pretty geometry theorems with proofs, as other important topics have replaced such old-fashioned things. The present book hopes to remedy this in some measure.

What's so special about geometry? We'll elaborate this below, but essentially:

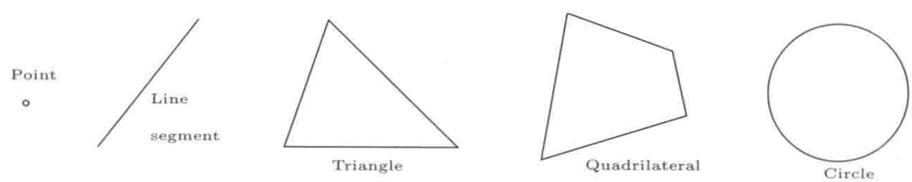
(1) It showed that mathematics involves not just *numbers*, but *pictures* (which is how many mathematicians *think* when creating new maths).
(2) It convinced the student that mathematics is beautiful and not boring. On the contrary, doing Mathematics can be *enjoyable*!
(3) Besides all these lovely things, it taught students what a "proof" is, and prepared high school graduates for university level mathematics.

The aim of this book is to cover the basics of the wonderful subject of Planar Geometry, at high school level, requiring no prerequisites beyond arithmetic, and hopefully to convey the sense of joy which I had when I was taught geometry.

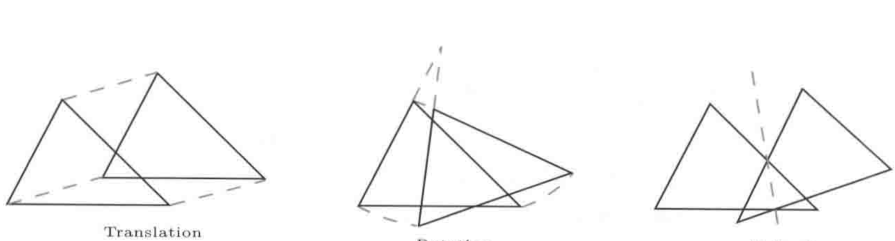
While one might argue about the use of teaching such outdated things, and contrast this with teaching other useful things such as Cartesian geometry, algorithms and so on, surely, as mentioned in item (3) above, there is one feature of Euclidean plane geometry which makes learning it worthwhile: it trains the student in understanding proofs, and also in devising one's own proofs: in other words, it inculcates the very spirit of Mathematics!

What is Planar Geometry?

Planar figures are figures in the plane, such as triangles, quadrilaterals and circles.



By *Planar Geometry*, we mean a study of planar figures and their geometrical properties. By a *geometric property* of a planar figure, we mean one which doesn't change under a "rigid" motion (that is motion that does not distort the figure: examples of such motion are rotation, translation and reflection). Examples of geometric properties that don't change are distances and angles.



Why study Planar Geometry?

Planar Geometry in some sense marks the birth of Mathematics as it is practiced in modern times. Often it is considered as the first historical treatise (the works of Euclid) in which the rules of doing Mathematics were set out in a systematic manner. If one asks: What is Mathematics? Then the following is a rough answer:

> Mathematics is a subject in which we make up definitions and prove things about the defined objects in a logical manner.

Thus Mathematics can be likened to a game, like Chess, in which there are:

(1) objects (the chess board, chess pieces),
(2) rules of the game (how the chess pieces move etc.),
(3) the play (when two players actually play the game).

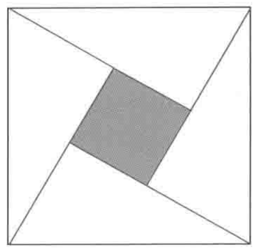
In a similar manner in Mathematics, there are:

(1) objects (definitions of mathematical objects),
(2) rules (mathematical logic),
(3) the play (making up theorems, that is, true statements about the objects and proving these truths).

This endeavor of doing mathematics is brought out very clearly for a school student by studying Planar Geometry à la Euclid, since

(1) The objects in Planar Geometry are points, lines, angles, triangles, quadrilaterals and circles, and these are quite easy to understand since they are concrete, and visual.
(2) The rules of the game are self-evident truths which are easy to accept, since they are again visual. These rules are also just a few in number and can be stated succinctly. Thus the logical structure of the proof becomes extremely clear to the student, when one gives reasons for the steps in the proof (such as usage of the Parallel Postulate, SAS Congruency Rule etc., which we will soon learn).
(3) The theorems in Planar Geometry, and their proofs are particularly beautiful and are ideal to convey the beauty of Mathematics to the uninitiated person. Indeed, prior to studying geometry, a school student's exposure is mostly limited to arithmetic, mostly with no "soul" or sense of "play". This changes drastically with Planar Geometry, since everything is visual, so the mind's eye can "see", and also play (by drawing and trying this and that, guessing, experiencing an "aha!"

moment, etc.). For example, in Exercise 3.27, we will encounter the following "proof without words", of Pythagoras's Theorem, saying that the square of the length of the biggest side in a right angled triangle is equal to the sum of the squares of the other two sides.



What will we learn in this book?

There are 5 chapters in the book:

(1) Geometrical figures
(2) Congruent triangles
(3) Quadrilaterals
(4) Similar triangles
(5) Circles

Perhaps the reader has encountered some or all of these objects before, and so might have some feeling of what is in store while reading this book. But besides this seemingly innocent backdrop of learning about planar geometry, the book has some ulterior motives:

(1) to learn drawing pictures, and realizing that Mathematics is not just about numbers, but can be very visual with pictures;
(2) that Mathematics can be fun: rather than being a monotonous process of carrying out some algorithm (like long division, calculation of percentages or solving a quadratic equation), it can be a creative process (where in order to construct a mathematical proof, one has to try out different possible ideas, get inspiration out of the blue, and sense exhilaration at solving an interesting problem (=puzzle) oneself, akin to solving an entire crossword puzzle, or winning a chess game, etc.;
(3) to teach what constitutes a proof (and en route instill rules of mathematical logic such as proof by contradiction, equivalent statements, converse of a statement, necessary and sufficient conditions, etc.);

(4) to develop mathematical maturity, which can be a combination of all of the above and more: when one knows what it means to "do Mathematics";

(5) to learn problem solving by doing the exercises, in which one can develop solution strategies. For example, learning to understand a mathematical problem by writing what is given and what is asked, finding a possible solution technique by asking if there is a simpler related problem which can be solved, or experimenting with the given data in the problem, considering extreme cases, and so on. Making informed and useful *guesses* in the form of possible constructions, or for example making claims about the equality of some pertinent angles or lengths or about perpendicularity in the picture, etc. Learning to subconsciously recognizing when one gets the key idea solving the problem, and then planning and writing the solution in a logical manner.

Funnily enough, no effort will be needed to specifically devote any energy on the above aims, as these will be *automatically* imparted when one practices Geometry!

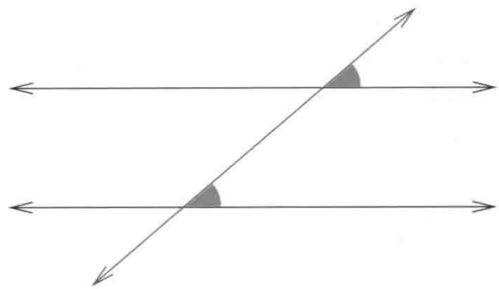
How did Planar Geometry arise?

The word "geometry" is derived from the Greek word "geo" meaning "Earth", and "metron" meaning "measurement".

$$\text{Geo} + \text{metron} = \text{Geometry}.$$

Historically Geometry seems to have arisen from a need to measure land. This is imaginable, since if a flood swept away the demarcations between plots of cultivable land, there would have been a need to redraw boundaries, and presumably this could have given rise to specific problems in planar geometry, leading to advancement in the subject over time. Other reasons could have been architecture and astronomy.

Historians of Mathematics claim that while there's evidence to show that ancient civilizations, such as the Indus Valley Civilization, the ancient Egyptians, and the Babylonians, knew geometry, the subject of Planar Geometry was developed *systematically* by the Greeks, and culminated around 300 BC with the works of Euclid, called *Elements*, in thirteen volumes. In these works, a new manner of thinking was introduced in geometry, in which geometrical results were proved by starting from an initial small collection of self-evident assumptions called *axioms*: for example when a line intersects two parallel lines, then the corresponding angles are equal:



Then other results in geometry, such as Pythagoras's Theorem, were deduced from these fundamental axioms by a process of logical reasoning. These derived new truths were called *theorems*, and the body of reasoning associated with a theorem was called its *proof*.

Who is the book for?

The book is written at high school level or beginning university level, but can be read by anyone interested in the subject.

How should the book be read?

The book follows the "Definition-Theorem-Proof-Exercise" format of Mathematics. The exercises are an integral part of studying this book. They are a combination of elementary ones (meant for understanding the definitions or simple usages of the theorems learnt), and those which are more challenging (sometimes indicated by an asterisk $(*)$), involving a careful and elaborate use of the theorems. The student should feel free to skip exercises which seem particularly challenging at the first instance, and return back to them now and again. Occasionally, the exercises also treat topics which are discussed subsequently in the text, and so they can be postponed until the relevant stuff has been covered. Although detailed solutions are provided, the student should not be tempted to consult the given solution too soon. To this end, a section on helpful hints is included, in order to give a nudge in the right direction. However, the solutions stem from the author's own point of view, and the reader might have an alternative solution.

The book does not aim to be encyclopedic, and there are several omissions. Nevertheless the hope is that, with the "Less is more ($\cdots$ provided it is pretty $\cdots$)" dictum I have tried to follow in this book, the uninitiated reader enjoys this book to a sufficient degree in order to get hooked, and learn more!

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I have made use of material from many diverse sources such as mathematics olympiad problem sets, mathematics education journals such as *The Mathematical Magazine* and *The Mathematical Gazette*, and online resources on topics in geometry. These are listed in the bibliography and at the end of chapters in the "Notes" sections. No claim to originality is made in case there is a missing reference.

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Amol Sasane
Lund, 2015

PLAIN PLANE GEOMETRY