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Characters and Blocks of Finite Groups

Gabriel Navarro

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Para Javier, Gabo, Nacho e Isabel

Preface

This set of notes grew out of a course that I gave at Ohio University in the spring of 1996. My aim was to give graduate students who were familiar with ordinary character theory an introduction to Brauer characters and blocks of finite groups.

To do that I chose an objective: the Glauberman Z^*-theorem. This theorem gives an excellent excuse for introducing modular representation theory to students interested in groups. Glauberman's outstanding result is one of the major applications of the theory to finite groups. However, to be able to prove it, one needs to proceed from the very basic facts to the three main theorems of R. Brauer.

In Chapter 1, I prove what is absolutely necessary to get started. Assuming that the students have already had a course on ordinary characters, I use this chapter to remind them of some familiar ideas while introducing some new ones.

In Chapter 2, I introduce Brauer characters (in the same spirit as in the book of M. Isaacs) and develop their basic properties.

In Chapter 3, I introduce blocks and, in Chapter 4, Brauer's first main theorem is given. The second main theorem is proven in Chapter 5 and its proof is a new "elementary" proof by Isaacs based on work by A. Juhász and Y. Tsushima. The third main theorem is given in a very general form and its proof is due to T. Okuyama. Once the third main theorem has been proven, we are ready for the Z^*-theorem.

After Glauberman's theorem is completed, I include Chapter 8 on the basic behaviour of Brauer characters. Blocks and Brauer characters of p-solvable groups are studied in Chapter 10.

The relationship between blocks and normal subgroups (which is needed in Chapter 10, but not for the Z^*-theorem) is covered in Chapter 9.

Finally, in Chapter 11, I develop one of the highlights of the whole theory: the description of the p-blocks of the groups with a Sylow p-subgroup of order p.

When writing down this set of notes, I could not resist introducing several topics which were not necessary to accomplish my objective, but which have interest on their own. Perhaps some of them may be taught if time allows it.

For many years, modular representation theory of finite groups was developed only through the incredible talent of Richard Brauer. I take this opportunity to express my deepest admiration for his work.

These notes would not have been possible without the help of Martin Isaacs, to whom I am very much indebted. Thomas Keller read the complete set of notes which have thus benefited from his comments. Chris Puin helped me with the English.

I also extend my thanks to G. Glauberman, M. Lewis, J. Muñoz, F. Pérez Monasor, L. Sanus, W. Willems and T. Wolf.

Valencia
December, 1997

Contents

1. Algebras

We will assume that all of our rings have an identity. If R is a ring, an abelian group M is a **left** R**-module** if for every $r \in R$ and $m \in M$, there is a unique element $rm \in M$ such that

$$r(a+b) = ra + rb\,,$$

$$(r+s)a = ra + sa\,,$$

$$(rs)a = r(sa)\,,$$

$$1_R a = a$$

for all $r, s \in R$ and $a, b \in M$. In the same way, but multiplying on the right, we define a **right** R**-module**.

If M and N are left R-modules, a map $f : M \to N$ is R**-linear** if f is additive and $f(rm) = rf(m)$ for all $m \in M$ and $r \in R$.

(1.1) DEFINITION. Suppose that R is a commutative ring and suppose that A is a left R-module. If A also is a ring such that

$$(ra)b = r(ab) = a(rb)$$

for all $r \in R$ and all $a, b \in A$, we say that A is an R**-algebra**.

When we think of R-algebras, we have two important examples in mind: $\mathrm{Mat}(n, R)$, the R-algebra of $n \times n$ matrices with entries in R and, for every finite group G, the **group algebra**

$$RG = \{\sum_{g \in G} a_g g \mid a_g \in R\}$$

with the multiplication of G extended linearly to RG. (In fact, representation theory studies the homomorphisms between RG and $\mathrm{Mat}(n, R)$.)

Also, if A is any ring and R is a subring of $\mathbf{Z}(A) = \{a \in A \mid ax = xa$ for all $x \in A\}$ (with the identity of A inside R), then A is an R-algebra.

If A and B are R-algebras, an **algebra homomorphism** is an R-linear, multiplicative map $f : A \to B$ such that $f(1_A) = 1_B$.

For the rest of this chapter, A is an R-algebra.

(1.2) DEFINITION. A left R-module V is said to be an A-**module** if V is a right A-module (A considered as a ring) such that for all $v \in V$, $r \in R$ and $a \in A$, we have that

$$(rv)a = r(va) = v(ra)\,.$$

One of the most important examples of an A-module is A itself with right multiplication. This is usually called the **regular** A-module.

If V is an A-module, a subgroup W of V is an A-**submodule** if $wa \in W$ for all $w \in W$ and $a \in A$. Notice that A-submodules are necessarily R-submodules since $rv = v(r1_A)$ for $r \in R$ and $v \in V$. Observe that the A-submodules of the regular A-module are the right ideals of A.

If W is an A-submodule of V, then V/W is an A-module via

$$(v + W)a = va + W$$

for $v \in V$ and $a \in A$.

(1.3) DEFINITION. We say that a nonzero A-module V is **simple** if its only A-submodules are 0 and V. (It is also common to say, in this case, that V is **irreducible**.)

If V and W are A-modules, an additive map $f : V \to W$ such that

$$f(va) = f(v)a$$

for all $v \in V$ and $a \in A$ is an A-**homomorphism** of modules. A bijective A-homomorphism is an **isomorphism** and we write $V \cong W$ in this case.

Notice that A-homomorphisms are necessarily R-linear since $f(rv) = f(v(r1_A)) = f(v)(r1_A) = rf(v)$ for $r \in R$ and $v \in V$.

If $f : V \to W$ is an A-homomorphism, then $\ker(f) = \{v \in V \mid f(v) = 0\}$ and $\mathrm{Im}(f)$ are A-submodules of V and W, respectively. Also, the map $v + \ker(f) \mapsto f(v)$ defines an isomorphism $V/\ker(f) \cong \mathrm{Im}(f)$.

If V and W are A-modules, we write $\mathrm{Hom}_A(V, W)$ for the abelian group of all A-homomorphisms $V \to W$. If $r \in R$ and $f \in \mathrm{Hom}_A(V, W)$, then $\mathrm{Hom}_A(V, W)$ is a left R-module via $(rf)(v) = rf(v)$ for $v \in V$. The set of all A-homomorphisms $V \to V$ is denoted by $\mathrm{End}_A(V)$. It is easy to check that $\mathrm{End}_A(V)$ is an R-algebra. Furthermore, R and $R1_V = \{r1_V \mid r \in R\}$ may be identified whenever R is a field.

(1.4) LEMMA (Schur). *Suppose that V and W are simple A-modules. Then every nonzero A-homomorphism $f : V \to W$ is invertible. As a consequence, if R is an algebraically closed field and $\dim_F(V)$ is finite, then $\mathrm{End}_A(V) = R$.*

Proof. Suppose that $f : V \to W$ is nonzero. Since $\ker(f)$ and $\mathrm{Im}(f)$ are A-submodules of V and W, respectively, it follows that $\ker(f) = 0$ and $\mathrm{Im}(f) = W$. Then f is bijective. To prove the latter assertion, we choose $0 \neq \lambda \in R$, an eigenvalue of f. Then $f - \lambda 1_V \in \mathrm{End}_A(V)$ is not invertible and therefore $f = \lambda 1_V$, by applying the first part. ∎

Sometimes, we use the fact that if R is an algebraically closed field and $f : V \to W$ is a nonzero A-homomorphism between two simple finite dimensional A-modules, then $\mathrm{Hom}_A(V, W) = \{rf \mid r \in R\}$. This easily follows from the second part in Schur's lemma.

If I is an ideal (we always mean double sided) of A, it is straightforward to check that A/I is an R-algebra.

The **annihilator** of an A-module V is $\mathrm{ann}(V) = \{a \in A \mid va = 0$ for all $v \in V\}$. This is an ideal of A, and notice that we may view V as an $(A/\mathrm{ann}(V))$-module.

We define the **Jacobson radical** of an R-algebra A to be the intersection of all $\mathrm{ann}(V)$ where V runs over all the simple A-modules. It is denoted by $\mathbf{J}(A)$, and certainly it is an ideal of A.

The next result tells us where to find the simple A-modules.

(1.5) THEOREM. *If A is an R-algebra and V is a simple A-module, then there exists a maximal right ideal I of A such that V and A/I are isomorphic. In fact, $\mathbf{J}(A)$ is the intersection of all maximal right ideals of A.*

Proof. If $0 \neq v \in V$, then $v \in vA = \{va \mid a \in A\}$. Thus, $vA = V$ since vA is a nonzero A-submodule of V. Now, the map $a \mapsto va$ from A onto V is an A-homomorphism of A-modules. Since $I = \mathrm{ann}(v) = \{a \in A \mid va = 0\}$ is the kernel of the map, A/I is isomorphic to V. The fact that V is simple makes I a maximal right ideal of A. Now, if J is the intersection of all maximal right ideals of A, we have that

$$J \subseteq \bigcap_{v \in V} \mathrm{ann}(v) = \mathrm{ann}(V)$$

and thus $\mathbf{J}(A)$ contains J. Now, if L is any maximal right ideal, then A/L is a simple A-module. Also, $\mathrm{ann}(A/L) \subseteq \mathrm{ann}(1 + L) = L$. Hence, $\mathbf{J}(A) \subseteq L$ for all such L. Thus $\mathbf{J}(A) \subseteq J$ and the proof of the theorem is completed. ∎

If $\mathbf{J}(A)$ is the unique maximal ideal of A, we say that A is **local**.

There is a useful fact about the elements of the Jacobson radical which will be used later on.

(1.6) THEOREM. *If A is an R-algebra and $a \in \mathbf{J}(A)$, then $1 - a$ is invertible.*

Proof. If $(1-a)A < A$, then the right ideal $(1-a)A$ is contained in some maximal right ideal M of A. In this case, since $\mathbf{J}(A) \subseteq M$, we have that $a \in M$ and we conclude that $1 \in M$, a contradiction. Therefore, we see that $(1-a)A = A$. Thus, we may find $1-b \in A$ such that $(1-a)(1-b) = 1$. We just need to prove that $(1-b)(1-a) = 1$. Since $(1-a)(1-b) = 1$, we see that $b = a(b-1) \in \mathbf{J}(A)$. Hence, by the same reasoning as before, $1-b$ has a right inverse, say c. Now, $1 - a = (1-a)((1-b)c) = ((1-a)(1-b))c = c$ and therefore $1-b$ is a left and right inverse of $1-a$, as required. ∎

If V is an A-module, $W \subseteq V$ and I is a right ideal of A, then WI denotes the additive subgroup of V generated by all the products wx with $w \in W$ and $x \in I$. Notice that WI is an A-submodule of V. By repeated application of this definition (with $V = A$), we can define I^n for every positive integer n. The right ideal I is **nilpotent** if there is an n with $I^n = 0$. (Note that $I^n = 0$ if and only if every product of n elements of I is zero.)

An A-module V is **finitely generated** if there exist $v_1, \ldots, v_n \in V$ such that

$$V = v_1 A + \ldots + v_n A .$$

(1.7) LEMMA (Nakayama). *Suppose that W is an A-submodule of V such that V/W is finitely generated over A. If $V = W + V\mathbf{J}(A)$, then $V = W$.*

Proof. It suffices to show the lemma for the case $W = 0$ and afterwards to apply it to V/W. So we have that V is a finitely generated A-module such that $V\mathbf{J}(A) = V$ and we wish to prove that $V = 0$. If $V \neq 0$, let $X \neq \emptyset$ be a minimal A-generating subset of V. Now,

$$V = V\mathbf{J}(A) = (\sum_{x \in X} xA)\mathbf{J}(A) = \sum_{x \in X} x\mathbf{J}(A) .$$

If $y \in X$, then we may write

$$y = \sum_{x \in X} x a_x ,$$

where $a_x \in \mathbf{J}(A)$. Now,

$$y(1 - a_y) = \sum_{x \in X - \{y\}} x a_x$$

and thus, by applying Theorem (1.6), we have that

$$y = \sum_{x \in X - \{y\}} x a_x (1 - a_y)^{-1} .$$

Therefore, $X - \{y\}$ generates V over A which contradicts the minimality of X. ∎

From now until the end of this chapter, we assume that $R = F$ is a field. Hence, from now on, every F-algebra is a vector space over F. We will assume that A and, in general, every A-module have finite dimension over F. Notice that, in this case, we may also assume that $F \subseteq \mathbf{Z}(A)$ since the map $f \mapsto f 1_A$ is an injective ring homomorphism from F into $\mathbf{Z}(A)$.

(1.8) THEOREM. *Suppose that A is an F-algebra. Then $\mathbf{J}(A)$ is the unique maximal nilpotent right ideal of A. Moreover,*

$$\mathbf{J}(\mathbf{Z}(A)) = \mathbf{J}(A) \cap \mathbf{Z}(A) .$$

Proof. We have that $\mathbf{J}(A)^n$ is an F-subspace of A, and thus it is finitely generated over A (since A contains F). By Nakayama's lemma (1.7), we have that $\mathbf{J}(A)^{n+1}$ is smaller than $\mathbf{J}(A)^n$, if this is nonzero. Hence, since the dimension of A is finite, we see that $\mathbf{J}(A)$ is necessarily nilpotent. Now, if I is a nilpotent right ideal of A and V is a simple A-module, then $VI = 0$ or $VI = V$ since VI is an A-submodule of V. If $VI = V$, then $VI^2 = (VI)I = VI = V$ and, in general, $VI^n = V$. But this is impossible because there is an integer m with $I^m = 0$. Thus, $VI = 0$ for all simple A-modules V and hence, $I \subseteq \mathbf{J}(A)$, as desired.

Finally, $\mathbf{J}(A) \cap \mathbf{Z}(A)$ is a nilpotent ideal of $\mathbf{Z}(A)$ and therefore, by the first part, it is contained in $\mathbf{J}(\mathbf{Z}(A))$. Now, let $z \in \mathbf{J}(\mathbf{Z}(A))$. Since $\mathbf{J}(\mathbf{Z}(A))$ is nilpotent and commutes with the elements of A, we have that zA is a nilpotent right ideal of A. Then $z \in \mathbf{J}(A)$. This proves that $\mathbf{J}(\mathbf{Z}(A)) \subseteq \mathbf{J}(A) \cap \mathbf{Z}(A)$. ∎

(1.9) DEFINITION. If A is an F-algebra, we say that A is **semisimple** if $\mathbf{J}(A) = 0$. Also, we say that A is **simple** if it has no proper (two sided) ideals.

Since A and $A/\mathbf{J}(A)$ have the same set of simple A-modules, it follows that $A/\mathbf{J}(A)$ is semisimple.

An A-module V is said to be **completely reducible** if it is the direct sum of simple A-submodules. (It is also common in this case to say that V is **semisimple**.) In fact, there is no difference between completely reducible modules and modules which may be written as a sum (not necessarily direct) of simple submodules.

(1.10) LEMMA. *Let V be an A-module and suppose $V = \sum V_i$, where the V_i's are simple submodules. Then V is the direct sum of some of the V_i's.*

Proof. Since V has finite dimension, we let W be an A-submodule of V maximal with respect to the property that W is the direct sum of some of the V_i's. If W is proper, then there exists a V_j not contained in W. But then, since V_j is simple, we have that $V_j \cap W = 0$. Then $W + V_j$ is a direct sum, which contradicts the maximality of W. ∎

More interesting is the next result.

(1.11) THEOREM. *If A is an F-algebra and V is an A-module, then the following conditions are equivalent.*

(a) V is completely reducible.

(b) If U is an A-submodule of V, then there is an A-submodule W such that $V = U \oplus W$.

Proof. Write $V = \sum V_i$, where the V_i's are simple submodules, and suppose that U is an A-submodule of V. Since V has finite dimension, let W be an A-submodule of V maximal such that $U + W = U \oplus W$. If $U + W$ is proper, then there is some V_j not contained in $U + W$. Since V_j is simple, $V_j \cap (U + W) = 0$. Therefore, $U + (V_j + W) = U \oplus (V_j + W)$, which contradicts the maximality of W. This proves that (a) implies (b).

Assume (b) and, since V is finite dimensional, let U be an A-submodule of V maximal such that U is a sum of simple A-submodules. By hypothesis, there is an A-submodule W of V such that $V = U \oplus W$. If $W \neq 0$, since V is finite dimensional, we may find W_0, a simple submodule of V inside W. Then $U + W_0 > U$, which contradicts the maximality of U. Hence, $U = V$ is completely reducible. ∎

(1.12) COROLLARY. *Suppose that V is a completely reducible A-module. If U is an A-submodule of V, then U and V/U are completely reducible.*

Proof. By Theorem (1.11), we have that V/U is isomorphic to a submodule of V. Hence, it suffices to show the first part. If W is an A-submodule of U, again by Theorem (1.11) we know that there exists an A-submodule W_0 of V such that $V = W \oplus W_0$. Then $U = W \oplus (U \cap W_0)$. This proves that U is completely reducible. ∎

If $V_1, \ldots, V_n$ are A-modules, we may form the **external direct sum** of $V_1, \ldots, V_n$, which is denoted by $V_1 \oplus \ldots \oplus V_n$, by setting $V_1 \oplus \ldots \oplus V_n = V_1 \times \ldots \times V_n$ with the action

$$(v_1, \ldots, v_n)a = (v_1 a, \ldots, v_n a)$$

for $v_i \in V_i$ and $a \in A$. It is clear that if V_i is a simple A-module for all i, then $V_1 \oplus \ldots \oplus V_n$ is a completely reducible A-module.

(1.13) THEOREM. *Suppose that A is an F-algebra. Then A is semisimple if and only if every A-module is completely reducible.*

Proof. Assume first that every A-module is completely reducible. If we consider A, the regular A-module, by hypothesis we have that $A = \sum_i I_i$ is a sum of minimal right ideals. Hence, $\mathbf{J}(A) = A\mathbf{J}(A) = 0$ since $\mathbf{J}(A)$ annihilates the simple A-module I_i for all i.

Assume now that A is semisimple. First, we prove that the regular A-module A is completely reducible. To do that, we claim that there exist maximal right ideals $M_1, \ldots, M_n$ of A such that

$$\bigcap_{j=1}^{n} M_j = 0\,.$$

If this is the case, the map $a \mapsto (a+M_1, \ldots, a+M_n)$ maps A isomorphically into a submodule of the completely reducible A-module $A/M_1 \oplus \ldots \oplus A/M_n$. Then, by Corollary (1.12), A is completely reducible. To prove the claim, among the subspaces L of A which are intersections of a finite number of maximal right ideals, we choose L of minimal dimension. If $L \neq 0$, then L is not contained in $\mathbf{J}(A) = 0$. Since $\mathbf{J}(A)$ is the intersection of all the maximal right ideals of A (Theorem (1.5)), we have that there exists a maximal right ideal M such that $L \cap M < M$. This contradicts the choice of L and proves the claim.

Now, write $A = \sum_i I_i$ as a sum of minimal right ideals of A. If V is an A-module and $\mathcal{B}$ is an F-basis of V, we have that

$$V = \sum_{v \in \mathcal{B}, i} vI_i\,.$$

Since the map $I_i \to vI_i$ given by $x \mapsto vx$ is a surjective A-homomorphism and I_i is a minimal right ideal, it follows that the kernel of the map is I_i or zero. Hence, vI_i is isomorphic to I_i or 0. Therefore, V is a sum of simple A-submodules, as required. ∎

(1.14) COROLLARY. *If A is a semisimple F-algebra and B is an ideal of A, then the F-algebra A/B is semisimple.*

Proof. If V is an (A/B)-module, then V is an A-module with $va = v(a+B)$ for $v \in V$ and $a \in A$. Hence, V is a sum of simple A-submodules. Since $VB = 0$, these are also simple (A/B)-submodules of V. Now, Theorem (1.13) applies. ∎