SEMINAR ON DIFFERENTIAL GEOMETRY

EDITED BY

SHING-TUNG YAU

ANNALS OF MATHEMATICS STUDIES
PRINCETON UNIVERSITY PRESS

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PRINCETON UNIVERSITY PRESS

AND

UNIVERSITY OF TOKYO PRESS

PRINCETON, NEW JERSEY

1982

Published in Japan exclusively by
University of Tokyo Press;
In other parts of the world by
Princeton University Press

Printed in the United States of America
by Princeton University Press, 41 William Street
Princeton, New Jersey

Library of Congress Cataloging in Publication data will
be found on the last printed page of this book

INTRODUCTION

In the academic year 1979-80, the Institute for Advanced Study and the National Science Foundation sponsored special activities in differential geometry, with particular emphasis on partial differential equations. In this volume, we collect all the papers which were presented in the seminars of that special program. Since there were many papers presented in the areas of closed geodesics and minimal surfaces, all the papers in these subjects have been collected in a separate volume. We would like to thank all the speakers for their enthusiastic participation and their cooperation in writing up their talks. We would also like to thank the National Science Foundation for supporting this special year.

SHING-TUNG YAU

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Seminar on Differential Geometry

SURVEY ON PARTIAL DIFFERENTIAL EQUATIONS IN DIFFERENTIAL GEOMETRY

Shing-Tung Yau*

In these talks, we are going to survey some analytic methods in differential geometry. The basic tools will be partial differential equations while the basic motivation is to settle problems in geometry or subjects related to geometry such as topology and physics. We shall order our exposition according to the nonlinearity of the partial differential equations that are involved in the geometric problems. It should be emphasized that these equations are related to each other in an intriguing manner, the major reason being that all these equations serve the same purpose of understanding geometric phenomena.

It is obvious that nonlinear equations are more complicated than linear equations and coupled systems of equations are more complicated than scalar valued equations. However, we should bear in mind that the understanding of linear equations is of fundamental importance in understanding nonlinear equations.

(I) Scalar Equations.

(A) Linear equations.

The basic linear operator in differential geometry is the Laplace-Beltrami operator $\Delta = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial}{\partial x^j}\right)$ where $\sum g_{ij} dx^i dx^j$ is the

*I would like to thank the typing staff at the Institute for Advanced Study for their usual excellent work, and Robert Bartnik for his assistance in compiling the bibliography.

Seminar on Differential Geometry
0-691-08268-5/82/000003-69$03.45/0 (cloth)
0-691-08296-0/82/000003-69$03.45/0 (paperback)
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metric and $g = \det(g_{ij})$. Associated with this operator, we have the Laplace equation, the heat equation and the wave equation. All these linear operators are connected with the eigenfunctions. These are functions u so that $\Delta u = -\lambda u$ where λ is a constant. Besides these linear operators, we also have the linear operator associated with the bending of the surface. If we consider a motion of a surface in three space which preserves the metric up to the first order, the field of motion satisfies a linear equation. If the surface is a graph of some function, this equation can be interpreted as the linearized equation of the Monge-Ampère operator which will be discussed later. The linear equation arising in this way is rather complicated because it is of mixed type in general.

(B) Equations whose highest order term is linear.

The typical equation that appears has the form $\Delta u = F(x,u)$ where F is a given smooth function. When we deform a metric conformally, the equation has either the form $\Delta u = k_1 e^u + k_2$ or $\Delta u = k_1 u^p + k_2 u$ where p is a constant and k_1, k_2 are given functions.

(C) Quasilinear equations.

The most important quasilinear equation in geometry is the minimal surface equation which has the form

$$\sum_i \frac{\partial}{\partial x^i}\left(\frac{\frac{\partial u}{\partial x^i}}{\sqrt{1+|\nabla u|^2}}\right) = 0 .$$

Notice that the coefficients of the highest order term involve the first derivatives of the unknown. This is what happens for quasilinear equations in general.

(D) The Monge Ampère equation.

This equation is nonlinear even in the highest order term. It has the form $\det\left(\frac{\partial^2 u}{\partial x^i \partial x^j}\right) = F$ for u defined on a domain in R^n. If one studies complex analysis, one will study the equation $\det\left(\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) = F$. These equations are closely related to the study of the curvature of a manifold.

(II) Systems.

(A) Linear systems.

The most important linear systems are the systems of harmonic forms, the Dirac equation, and the $\bar{\partial}$-equation. The last is overdetermined when the dimension of the complex space is greater than one. It makes the system more rigid. The study of these systems is related to harmonic theory.

(B) Linear systems whose highest order term is linear.

The typical system is the system of harmonic maps between Riemannian manifolds. The other system is the Yang-Mills equation when we choose a suitable gauge.

(C) Quasilinear systems.

As in the scalar case, the most important quasilinear system is the system corresponding to minimal submanifolds.

(D) Systems associated to the isometric immersion of a Riemannian manifold into another manifold.

This is an underdetermined system and the most celebrated work was done by Nash [N1].

(E) Systems associated with a prescribed curvature tensor.

This may be considered as a generalization of the Monge-Ampère equation to systems. The most important system is the Einstein Field equation. The question is that given a tensor on the manifold, how do we find a metric on the manifold so that some part of the curvature tensor is the given tensor? In the case of the Einstein Field equation, we are given the energy-stress tensor and we are asked to find a metric whose normalized

Ricci tensor is this energy-stress tensor. If we are looking in the category of Lorentz metrics, the system is hyperbolic.

§1. *The isoperimetric, Poincaré and Sobolev inequalities*

We start with the most basic inequalities in analysis. These are the Poincaré and Sobolev inequalities. The Poincaré inequality can be derived from the Sobolev inequality while the Sobolev and isoperimetric inequalities are essentially equivalent.

The Poincaré inequality states that for any compact manifold M with boundary ∂M, there exists a constant $c > 0$ such that for any smooth function f which vanishes on ∂M,

$$(1.1) \qquad c \int_M f^2 \leq \int_M |\nabla f|^2 .$$

The Sobolev inequality states that there exists a constant $c' > 0$ such that for any smooth function f which vanishes on ∂M,

$$(1.2) \qquad c' \left(\int_M f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq \int_M |\nabla f| .$$

Here n is the dimension of M.

These inequalities are for functions satisfying Dirichlet boundary conditions. If we assume $\int_M f = 0$ instead of $f = 0$ on ∂M, then the inequalities (1.1) and (1.2) are still valid with different constants c and c' (which are independent of f). The condition $\int_M f = 0$ is usually called the Neumann condition.

It should be noted that the inequality (1.2) implies that for all $n > p > 1$:

$$(1.3) \qquad \frac{(n-p)c'}{p(n-1)} \left(\int_M |f|^{\frac{np}{n-p}} \right)^{\frac{n-p}{np}} \leq \left(\int_M |\nabla f|^p \right)^{\frac{1}{p}} .$$

This is a simple application of the Hölder inequality.

The Poincaré inequality and the Sobolev inequality are basic tools in the theory of partial differential equations. It turns out that the largest positive constant c in the Poincaré inequality is the smallest eigenvalue for the Laplacian acting on functions satisfying the Dirichlet boundary condition or the Neumann condition depending on the assumption $f = 0$ on ∂M or $\int_M f = 0$. This is a consequence of the mini-max principle (see Courant-Hilbert [CH] Vol. I, page 399).

In fact, let H be the Hilbert space of functions f on M so that $\int_M |\nabla f|^2 < \infty$ and $f = 0$ on ∂M in case we are dealing with the Dirichlet boundary condition and $\frac{\partial f}{\partial \nu} = 0$ on ∂M in case we are dealing with the Neumann condition. Then the spectral theory says that we can find a countable orthonormal basis of H consisting of eigenfunctions f_i with

$$\Delta f_i = -\lambda_i f_i \tag{1.4}$$

and

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots .$$

The mini-max principle asserts that

$$\lambda_1 = \inf \left\{ \left(\int_M |\nabla f|^2 \right) \left(\int_M f^2 \right)^{-1} : f \in H \right\} \tag{1.5}$$

$$\lambda_i = \inf \left\{ \left(\int_M |\nabla f|^2 \right) \left(\int_M f^2 \right)^{-1} : f \in H \right.$$

and

$$\left. \int_M f f_j = 0 \quad \text{for} \quad j = 1, \cdots, i-1 \right\} .$$

This characterization of the eigenvalues of the Laplacian has been the basic tool to estimate the eigenvalues. It is especially effective for the first few eigenvalues.

It is also well known (see [Ch2]) that λ_1 is closely related to some constants arising in the isoperimetric inequality. Let

$$h_D(M) = \inf\left\{\frac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(\Omega)} : \Omega \text{ is a compact subdomain of } M\right\} \tag{1.6}$$

and

$$h_N(M) = \inf\left\{\frac{\mathrm{Vol}(H)}{\min(\mathrm{Vol}(M_1), \mathrm{Vol}(M_2))} : H \text{ is a hypersurface of } M \text{ which decomposes } M \text{ into } M_1 \text{ and } M_2\right\}. \tag{1.7}$$

Then by studying the level set of the first eigenfunction [Ch1], one can prove that

$$\lambda_1 \geq \frac{1}{4} h_D(M)^2 \tag{1.8}$$

for the first eigenvalue of the Dirichlet problem and

$$\lambda_1 \geq \frac{1}{4} h_N(M)^2 \tag{1.9}$$

for the first eigenvalue of the Neumann problem.

In this explicit form, (1.8) and (1.9) are due to Cheeger [Ch1]. A formula of this sort is still lacking for the Laplacian acting on differential forms. Such a formula will give a better understanding of nonlinear elliptic systems on a Riemannian manifold.

Inequalities (1.8) and (1.9) were used in [Y2] to give a lower estimate of λ_1 in terms of some more precise geometric data of M. C. Croke [Cr] was able to push [Y2] further by using an idea of Berger and Kazdan [Bes]. He was able to estimate the Sobolev constant for a compact Riemannian manifold by the same geometric data as in [Y2].

The Sobolev inequality is equivalent to the isoperimetric inequality. In fact, if Ω is a compact subdomain in M and if we choose f in (1.2) to be a function which approximates the characteristic function of Ω, then taking the limit one obtains

$$c'(\mathrm{Vol}\,\Omega)^{\frac{n-1}{n}} \le \mathrm{Vol}(\partial\Omega) \tag{1.10}$$

which is the isoperimetric inequality for domains in M.

One can prove (see [FF]) that if (1.10) holds for all compact subdomains of M, then (1.2) holds. Hence a demonstration of (1.2) is reduced to a demonstration of (1.10).

To state the isoperimetric inequality of Croke, we define a constant $\omega(\Omega)$. For each point $x \in \Omega$, let ω_x be the Lebesgue measure of the set of all unit vectors $v \in T_x(\Omega)$ such that the geodesic issuing from x with tangent v intersects $\partial\Omega$ and the geodesic segment from x to its first point of intersection with $\partial\Omega$ has minimal distance. The quantity $\omega(\Omega)$ is defined to be $\min_{x\in\Omega} \omega_x$.

In [Y2], we estimate $h(M)$ in terms of $\omega(M)$ and the diameter of M. Croke then proved

$$\mathrm{Vol}(\partial\Omega)^n \ge c_n\,\omega(\Omega)^{n+1}\,\mathrm{Vol}(\Omega)^{n-1} \tag{1.11}$$

where the equality holds if and only if $\omega(\Omega) = 1$ and Ω is isometric to a hemisphere of constant curvature.

If Ω is a subdomain of a simply connected complete manifold without conjugate points, then $\omega(\Omega) = 1$. If Ω is a compact subdomain of a general complete manifold M, then $\omega(\Omega)$ can be estimated as follows:

Suppose Ω is a subset of some geodesic ball $B(r)$ of radius r. For each $x \in \Omega$, we can consider the exponential map $\exp_x$ at x and its Jacobian $\sqrt{g}(x,y)$ at $y \in T_x(\Omega)$. As is well known ([BC]), an upper bound of $\sqrt{g}$ can be estimated in terms of the lower bound of the Ricci curvature of M. Using this function, we can estimate $\omega(\Omega)$ as follows:

$$\omega(\Omega) \ge \mathrm{Vol}(B(kr) - B(r)) \left(\sup_{x\in\Omega} \int_0^{(k+1)r} \sup_{\overline{x,y}=t} \sqrt{g}(x,y)\,t^{n-1}\,dt \right)^{-1} \tag{1.11}$$

where $\overline{x,y}$ denotes the distance between x and y and k is any number greater than one.

The proof of (1.11) is rather easy. One forms a cone by joining every point in $B(kr) - B(r)$ to $x \in \Omega$ by a shortest geodesic. These geodesics must intersect $\partial\Omega$. Hence ω_x must be not less than the solid angle formed by the cone. By computing the volume of this cone in terms of $\sqrt{g}$, we obtain the formula (1.11).

If we have information about the Ricci curvature of M, we can estimate $\sqrt{g}$ as follows: Let $(n-1)K$ be the lower bound of the Ricci curvature of M where $K \leq 0$. Then

$$\sqrt{g}(x,y) \leq \left(\frac{\sinh \sqrt{-K}\, r}{\sqrt{-K}\, r}\right)^{n-1} . \tag{1.12}$$

where $r = \overline{x,y}$.

In particular, if the Ricci curvature of M is nonnegative, then $\sqrt{g}(x,y) \leq 1$. Hence it follows easily from (1.11), and (1.12) that *for a complete manifold with nonnegative Ricci curvature such that* $\lim_{r\to\infty} \mathrm{Vol}(B(r))\, r^{-n} > 0$, *the isoperimetic inequality holds for all compact subdomains* Ω *of* M *where the constant is independent of* Ω. *As was pointed out before, this means that the Sobolev inequality holds for smooth functions with compact support for this class of manifolds.*

The Sobolev inequality for functions with compact support is, of course, very important. However, in applications, it is also very important to prove a Poincaré inequality or Sobolev inequality for functions without compact support. This type of inequality is much more subtle and is much more sensitive to the boundary of the domain under consideration. We mention an inequality of this type in the following.

Let $B(r)$ be the geodesic ball with radius r, and with fixed centre. Let $\beta > 0$ be chosen so that

$$\mathrm{Vol}((1-\beta)r) = \frac{3}{4} \mathrm{Vol}(B(r)) . \tag{1.13}$$

Then using the method of [Y2], one can prove that for $p \geq 1$,