Spline Functions: Basic Theory

Third Edition

Larry L. Schumaker

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SPLINE FUNCTIONS

IN MEMORY OF
STEPHEN ANTHONY SCHUMAKER

PREFACE

The theory of spline functions and their applications is a relatively recent development. As late as 1960, there were no more than a handful of papers mentioning spline functions by name. Today, less than 20 years later, there are well over 1000 research papers on the subject, and it remains an active research area.

The rapid development of spline functions is due primarily to their great usefulness in applications. Classes of spline functions possess many nice structural properties as well as excellent approximation powers. Since they are easy to store, evaluate, and manipulate on a digital computer, a myriad of applications in the numerical solution of a variety of problems in applied mathematics have been found. These include, for example, data fitting, function approximation, numerical quadrature, and the numerical solution of operator equations such as those associated with ordinary and partial differential equations, integral equations, optimal control problems, and so on. Programs based on spline functions have found their way into virtually every computing library.

It appears that the most turbulent years in the development of splines are over, and it is now generally agreed that they will become a firmly entrenched part of approximation theory and numerical analysis. Thus my aim here is to present a fairly complete and unified treatment of spline functions, which, I hope, will prove to be a useful source of information for approximation theorists, numerical analysts, scientists, and engineers.

This book developed out of a set of lecture notes which I began preparing in the fall of 1970 for a course on spline functions at the University of Texas at Austin. The material, which I have been reworking ever since, was expanded and revised several times for later courses at the Mathematics Research Center in Madison, the University of Munich, the University of Texas, and the Free University of Berlin. It was my original intent to cover both the theory and applications of spline functions in a single monograph, but the amount of interesting and useful material is so large that I found it impossible to give all of it a complete and comprehensive treatment in one volume.

This book is devoted to the basic theory of splines. In it we study the main algebraic, analytic, and approximation-theoretic properties of various spaces of splines (which in their simplest form are just spaces of piecewise polynomials). The material is organized as follows. In Chapters 1 to 3 background and reference material is presented. The heart of the book consists of Chapters 4 to 8, where polynomial splines are treated. Chapters 9 to 11 deal with the theory of generalized splines. Finally, Chapters 12 to 13 are devoted to multidimensional splines. For the practical-minded reader, I include a number of explicit algorithms written in an easily understood informal language.

It has not been my aim to design a textbook, *per se*. Thus throughout the book there is a mixture of very elementary results with rather more sophisticated ones. Still, much of it can be read with a minimum of mathematical background—for example calculus and linear algebra. With a judicious choice of material, the book can be used for a one-semester introduction to splines. For this purpose I suggest drawing material from Chapters 1 to 6, 8, and 12, with special emphasis on Chapters 4 and 5.

The notation in the book is quite standard. In order to keep the exposition moving as much as possible, I have elected to move most of the remarks and references to the end of the chapters. Thus each chapter contains sections with remarks and with historical notes. In these sections I have attempted, to the best of my ability, to trace the sources of the ideas in the chapter, and to guide the reader to the appropriate references in the massive literature.

I would like to take this opportunity to acknowledge some of the institutions and individuals who have been of assistance in the preparation of this book. First, I would like to thank Professor Samuel Karlin for introducing me to spline functions when I was his graduate student at Stanford in the early sixties. The Mathematics Research Center at the University of Wisconsin gratiously supported me at two critical junctures in the evolution of this book. The first was in 1966 to 1968 when the lively research atmosphere and the close contact with such experts as Professors T. N. E. Greville, M. Golomb, J. W. Jerome, and I. J. Schoenberg sharpened my interest in splines and taught me much about the subject. The support of the Mathematics Research Center again in 1973 to 1974 gave me a much needed break to continue work on the book.

In 1974 to 1975 I was at the Ludwig-Maximilians Universität in Munich. My thanks are due to Professor G. Hämmerlin for the invitation to visit Munich, and to the Deutsche Forschungsgemeinschaft for their support. Since January of 1978 I have been at the Free University of Berlin and the Hahn-Meitner Atomic Energy Institute. I am grateful to Professors K.-H.

Hoffmann and H.-J. Töpfer for suggesting and arranging my visit, and to the Humboldt Foundation of the Federal Republic of Germany for their part in my support. Finally, I would like to express my appreciation to the U.S. Air Force Office of Scientific Research and the Center of Numerical Analysis of the University of Texas for support of my research over the past several years.

Among the many colleagues and students who have read portions of the manuscript and made useful suggestions, I would especially like to mention Professors Carl deBoor, Ron DeVore, Tom Lyche, Charles Micchelli, Karl Scherer, and Ulrich Tippenhauer. The task of tracking down and organizing the reference material was formidable, and I was greatly assisted in this task by Jannelle Odem, Maymejo Moody Barrett, Nancy Jo Ethridge, Linda Blackman, and Patricia Stringer. Finally, I would like to thank my wife Gerda for her constant support, and for her considerable help in all stages of the preparation of this book.

LARRY L. SCHUMAKER

Britton, South Dakota

PREFACE TO THE 3RD EDITION

This book was originally published by Wiley-Interscience in 1981. A second edition was published in 1993 by Krieger. The two differ only in that a number of misprints were corrected. Both editions are now out of print. However, spline functions remain an active research area with important applications in a wide variety of fields, including some, such as Computer-Aided Geometric Design (CAGD) and Wavelets, which did not exist in 1981. This continued interest in the basic theory of splines was the motivation for preparing this third edition of the book.

There have been many developments in the theory of splines over the past twenty-five years. While it was not my intention of rewrite this book to cover all of these developments, David Tranah of Cambridge University Press convinced me that it would be useful to prepare a supplement to the book which gives an overview of the main developments with pointers to the literature. Tracking down this literature was a major undertaking, and more than 250 new references are included here. However, this is still far from a complete list. For an extended list, see the online bibliography at `www.math.vanderbilt.edu/~schumake/splinebib.html`. I include links there to a similar bibliography for splines on triangulations, and to the much larger spline bibliography in TeX form maintained by Carl de Boor and I.

Interpolation, approximation, and the numerous other applications of splines are not treated in this book due to lack of space. Consequently, I have elected not to discuss them in the supplement either, and the new list of references does not include any applied papers or books.

I would like to thank my many colleagues and friends who provided references to their recent work on splines. I am especially indebted to Carl de Boor, Oleg Davydov, Kirill Kopotun, and Tom Lyche for their comments on an early draft of the supplement. I am also grateful to Simon Foucart for a careful reading of the final version. Finally, my deepest appreciation to my wife Gerda for her patience over the many years it took to write this book and the companion book *Splines on Triangulations* (with M.-J. Lai, Cambridge University Press, 2007).

February, 2007 Larry L. Schumaker

CONTENTS

1
INTRODUCTION

The first three chapters of this book are devoted to background material, notation, and preliminary results. The well-prepared reader may wish to proceed directly to Chapter 4 where the study of spline functions *per se* begins.

§ 1.1. APPROXIMATION PROBLEMS

Functions are the basic mathematical tools for describing and analyzing many physical processes of interest. While in some cases these functions are known explicitly, very frequently it is necessary to construct approximations to them based on limited information about the underlying processes. Such approximation problems are a central part of *applied mathematics*.

There are two major categories of *approximation problems*. The first category consists of problems where it is required to construct an approximation to an unknown function based on some finite amount of data (often measurements) on the function. We call these *data fitting problems*. In such problems, the data are often subject to error or noise, and moreover, usually do not determine the function uniquely. Data fitting problems arise in virtually every branch of scientific endeavor.

The second main category of approximation problems arises from mathematical models for various physical processes. As these models usually involve operator equations that determine the unknown function, we refer to them as *operator-equation problems*. Examples include boundary-value problems for ordinary and partial differential equations, eigenvalue–eigenfunction problems, integro–differential equations, integral equations, optimal control problems, and so on. While there are many theoretical results on existence, uniqueness, and properties of solutions of such operator equations, usually only the simplest specific problems can be solved explicitly. In practice we will usually have to construct approximate solutions.

The most commonly used approach to finding approximations to unknown functions proceeds as follows:

1. Choose a reasonable *class of functions* in which to look for an approximation.
2. Devise an appropriate *selection scheme* (≡*approximation process*) for assigning a specific function to a specific problem.

The success of this approach depends heavily on the existence of convenient classes of approximating functions. To be of maximal use, a class $\mathcal{Q}$ of approximating functions should possess at least the following basic properties:

1. The functions in $\mathcal{Q}$ should be relatively smooth;
2. The functions in $\mathcal{Q}$ should be easy to store and manipulate on a digital computer;
3. The functions in $\mathcal{Q}$ should be easy to evaluate on a computer, along with their derivatives and integrals;
4. The class $\mathcal{Q}$ should be large enough so that arbitrary smooth functions can be well approximated by elements of $\mathcal{Q}$.

We have required property 1 because functions arising from physical processes are usually known to be smooth. Properties 2 and 3 are important because most real-world problems cannot be solved without the help of a high-speed digital computer. Finally, property 4 is essential if we are to achieve good approximations.

The study of various classes of approximating functions is precisely the content of *approximation theory*. The design and analysis of effective algorithms utilizing these approximation classes are a major part of *numerical analysis*. Both of these fields have a rich history, and a voluminous literature.

The purpose of this book is to examine in considerable detail some specific approximation classes—the so-called spline functions—which in the past several years have proved to be particularly convenient and effective for approximation purposes. Because of space limitations, we shall deal only with the basic theoretical properties of spline functions. Applications of splines to data fitting problems and to the numerical solution of operator equations will be treated in later monographs.

§ 1.2. POLYNOMIALS

Polynomials have played a central role in approximation theory and numerical analysis for many years. To indicate why this might be the case, we note that the space

$$\mathcal{P}_m = \left\{ p(x) : p(x) = \sum_{i=1}^{m} c_i x^{i-1}, \qquad c_1, \ldots, c_m, x \text{ real} \right\} \tag{1.1}$$

of *polynomials of order m* has the following attractive features:

1. $\mathcal{P}_m$ is a finite dimensional linear space with a convenient basis;
2. Polynomials are smooth functions;
3. Polynomials are easy to store, manipulate, and evaluate on a digital computer;
4. The derivative and antiderivative of a polynomial are again polynomials whose coefficients can be found algebraically (even by a computer);
5. The number of zeros of a polynomial of order m cannot exceed $m-1$;
6. Various matrices (arising in interpolation and approximation by polynomials) are always nonsingular, and they have strong sign-regularity properties;
7. The sign structure and shape of a polynomial are intimately related to the sign structure of its set of coefficients;
8. Given any continuous function on an interval $[a,b]$, there exists a polynomial which is uniformly close to it;
9. Precise rates of convergence can be given for approximation of smooth functions by polynomials.

We shall examine each of these assertions in detail in Chapter 3, along with a number of other properties of polynomials.

While this list tends to indicate that polynomials should be ideal for approximation purposes, in practice, it has been observed that they possess one unfortunate feature which allows for the possibility that still better classes of approximating functions may exist; namely,

10. Many approximation processes involving polynomials tend to produce polynomial approximations that oscillate wildly.

We illustrate this feature of polynomials in Section 3.6. It is a kind of *inflexibility* of the class $\mathcal{P}_m$.

§ 1.3. PIECEWISE POLYNOMIALS

As mentioned in the previous section, the main drawback of the space $\mathcal{P}_m$ of polynomials for approximation purposes is that the class is relatively inflexible. Polynomials seem to do all right on sufficiently small intervals, but when we go to larger intervals, severe oscillations often appear—particularly if m is more than 3 or 4. This observation suggests that in order to achieve a class of approximating functions with greater flexibility, we should work with polynomials of relatively low degree, and should divide up the interval of interest into smaller pieces. We are motivated to make the following definition:

DEFINITION 1.1. Piecewise Polynomials

Let $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$, and write $\Delta = \{x_i\}_0^{k+1}$. The set Δ partitions the interval $[a,b]$ into $k+1$ subintervals, $I_i = [x_i, x_{i+1})$, $i = 0, 1, \ldots, k-1$, and $I_k = [x_k, x_{k+1}]$. Given a positive integer m, let

$$\mathcal{P}\mathcal{P}_m(\Delta) = \begin{array}{l} \{f\text{: there exist polynomials} \\ p_0, p_1, \ldots, p_k \text{ in } \mathcal{P}_m \text{ with } f(x) = p_i(x) \\ \text{for } x \in I_i,\ i = 0, 1, \ldots, k\}. \end{array} \tag{1.2}$$

We call $\mathcal{P}\mathcal{P}_m(\Delta)$ the *space of piecewise polynomials* of *order m* with *knots* $x_1, \ldots, x_k$.

The terminology in Definition 1.1 is perfectly descriptive—an element $f \in \mathcal{P}\mathcal{P}_m(\Delta)$ consists of $k+1$ polynomial pieces. Figure 1 shows a typical example of a piecewise polynomial of order 3 with two knots.

While it is clear that we have gained flexibility by going over from polynomials to piecewise polynomials, it is also obvious that at the same time we have lost another important property—piecewise polynomial functions are not necessarily smooth. In fact, as shown in Figure 1, they can even be discontinuous. In most applications, the user would be happier if the approximating functions were at least continuous. Indeed, it is probably precisely this defect of piecewise polynomials which accounts for the fact that prior to 1960 they played a relatively small role in approximation theory and numerical analysis—for an historical account, see Section 1.6.

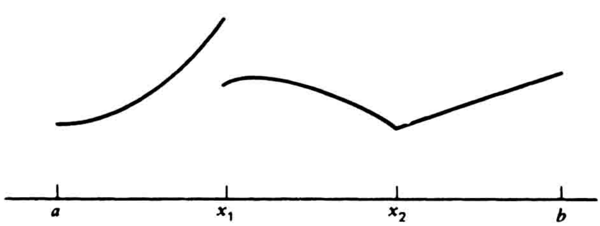

Figure 1. A quadratic piecewise polynomial.