APPLICATIONS OF DISCRETE AND CONTINUOUS FOURIER ANALYSIS

H. JOSEPH WEAVER

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To my sons,
Joeby, Colin, and Kirk

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PREFACE

Fourier analysis is used to describe the behavior of a remarkably wide range of scientific and engineering phenomena. Since its introduction by Daniel Bernoulli over two centuries ago, it has been studied and developed to the point where it is perhaps the most useful form of mathematical analysis available today.

In this text, I have divided Fourier analysis into three analogous topics. First is the Fourier series—a mapping that converts a (periodic) function to a sequence of Fourier series coefficients. Next is the Fourier transform which maps a function to another function. Finally, we have the discrete Fourier transform which maps one sequence to another. Both the Fourier series and Fourier transform mappings have wide application in the study of physical phenomena. On the other hand, the utility of the discrete Fourier transform lies in the fact that it can be used to digitally calculate the other two mappings.

My intention, when designing this text, was to produce a work that would enable the reader to understand the concept and properties of the Fourier mappings and then apply this knowledge to the analysis of real scientific and engineering problems. To accomplish this goal, I have divided the text into two portions. The first, consisting of Chapters 1–5, presents the definition and properties of the Fourier series, Fourier transform, and discrete Fourier transform. Since this is a text on applications, the presentation in this first section does not dwell upon the questions of existence and reciprocity of the mappings but rather the emphasis is placed on how their properties can be used to simplify their calculation and manipulation. Chapters 6–11 constitute the second portion which builds upon the material presented in the first portion and demonstrates various physical applications of Fourier analysis.

In very simple terms, Fourier analysis is the study of how general functions can be decomposed into linear combinations of the trigonometric sine and cosine functions. Chapter 1 discusses this concept of frequency content of a function from both heuristic and mathematical points of view.

Chapters 2, 3, and 4 discuss the Fourier series, Fourier transform, and discrete Fourier transform, respectively. In each of these chapters the basic definition is presented and then the various properties are discussed. The presentations are presented in an analogous fashion so that the similarities of the mappings are highlighted.

Chapter 5 presents a discussion of the digital calculation of both the Fourier series and Fourier transform by using the discrete Fourier transform. In this chapter, sampling theory is discussed from a practical point of view.

Chapter 6 discusses the concepts of the impulse response and transfer function of a system. These concepts from systems theory are used throughout the remaining applications chapters.

The remaining five chapters deal directly with applications to specific fields. Specifically, Chapter 7 discusses application of Fourier analysis to both mechanical and electrical systems as well as the one-dimensional wave equation. Chapter 8 deals with the physics of optical wave propagation and optical systems engineering. Chapter 9 deals with the accuracy of numerical analysis algorithms from a frequency domain point of view. Chapter 10 discusses applications of Fourier analysis to the solution of the heat, or diffusion, equation. Chapter 11 discusses basic applications of Fourier analysis to statistics and probability theory as well as a brief presentation of stochastic systems analysis.

This text is based upon a portion of the material that was used as course notes at the University of California's Lawrence Livermore National Laboratory. This course was offered as part of their continuing education program. I am very grateful to several of those students whose comments and suggestions were very helpful in developing the new manuscript. In particular, my thanks to B. J. McKinley, Karena McKinley, Henry Chau, and Dr. Jeff Richardson for their fine efforts. Very special thanks must go to Dr. Gary Sommargren, a good friend and colleague, for his expert advice, criticism, and encouragement throughout the development of this text. Finally, deepest gratitude to my wife Sue who gave up many evenings to help type and proof this manuscript.

H. JOSEPH WEAVER

Livermore, California
May 1983

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CHAPTER

1

THE CONCEPT OF FREQUENCY CONTENT

Fourier analysis, or frequency analysis, in the simplest sense is the study of the effects of adding together sine and cosine functions. This type of analysis has become an essential tool in the study of a remarkably large number of engineering and scientific problems. Daniel Bernoulli, while studying vibrations of a string in the 1750s, first suggested that a continuous function over the interval $(0, \pi)$ could be represented by an infinite series consisting only of sine functions. This suggestion was based on his physical intuition and was severely attacked by mathematicians of the day. Roughly 70 years later J. B. Fourier reopened the controversy while studying heat transfer. He argued, more formally, that a function continuous on an interval $(-\pi, \pi)$ could be represented as a linear combination of both sine and cosine functions. Still his conjecture was not readily accepted, and the question went unresolved for many years.

The purpose of this chapter is to present the concept of frequency content, or to say it another way, to study the effect of a linear combination of sine and cosine functions. To do this, we present both the definition and functional behavior of these basic building blocks (sine and cosine functions). At first, our approach is rather heuristic and the emphasis placed on the trigonometric derivation of these functions. From this derivation their functional behavior is demonstrated. Simple examples are used to illustrate how a combination of these functions is dependent upon both the amplitude and frequency of the individual trigonometric function components. Once the "physical" concepts have been presented, we present a concise mathematical definition of frequency content. Finally, we show how complex variable theory can be used to enhance our understanding and appreciation of the concept of frequency content.

TRIGONOMETRIC SINE AND COSINE FUNCTIONS

The fundamental building blocks of Fourier analysis are the sine and cosine trigonometric functions. Trigonometry literally means "the measurement of three angles" and was first used to study the relationship between the sides and angles of a triangle. Today, however, the trigonometric functions themselves have become central objects of study in the modern mathematical field called analysis. The mathematician uses sines and cosines to study other functions, whereas the engineer and scientist use them to study certain periodic phenomenon. Although the sine and cosine functions can be introduced from either point of view, we approach the subject from the classical or "triangle solving" method. This serves to present the more physical side of these functions from which their periodic nature and analytical properties can be demonstrated. The right triangle shown in Figure 1.1 will be our starting point. The relations sine,

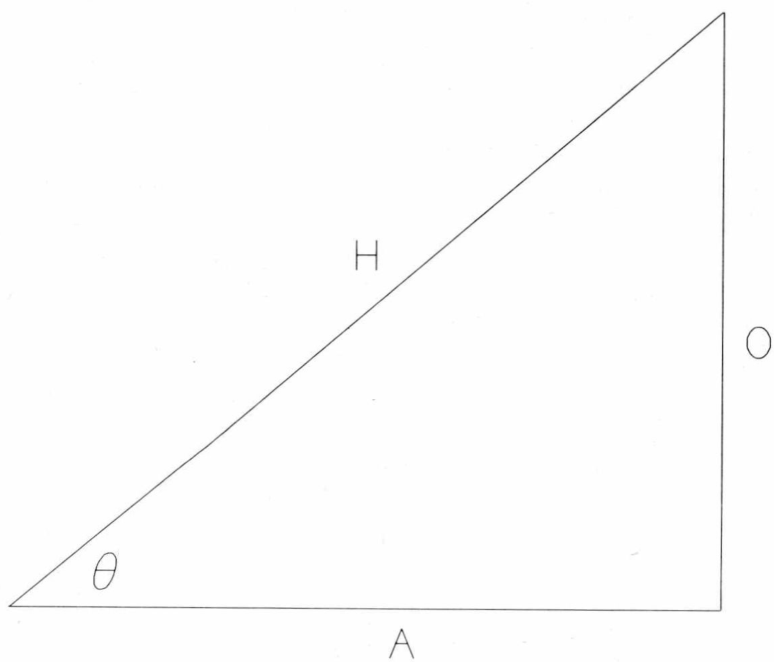

FIGURE 1.1. Trigonometric relations on a right triangle.

cosine, and tangent are defined on this triangle as follows:

(1.1a) $$\sin\theta = \frac{O}{H},$$

(1.1b) $$\cos\theta = \frac{A}{H},$$

(1.1c) $$\tan\theta = \frac{O}{A}.$$

Equations (1.1a)–(1.1c) are our basic definitions of the sine and cosine relations. These relations can be considered to give the measure of an angle in terms of the sides of a right triangle. The sine and cosine functions are dimensionless. The dimensions of the angle θ (which is discussed shortly) may be radians, gradians, degrees, or any other convenient measure one cares to define. Usually associated with trigonometry and the study of triangles are the units of degrees. However, we prefer radians. Let us now add to our repertoire the result of the well-known Pythagorean theorem which states

(1.2) $$H^2 = A^2 + O^2.$$

This equation can be used in conjunction with Equations (1.1) to derive many familiar results. For example, if we first solve Equations (1.1) for O and A in terms of sine and cosine and then square the resulting equations we obtain

(1.3) $$O^2 = H^2\sin^2\theta, \quad A^2 = H^2\cos^2\theta.$$

Substitution of these equations into Equation (1.2) and dividing by H^2 yields

(1.4) $$\sin^2\theta + \cos^2\theta = 1.$$

As a second example, let us divide Equation (1.1a) by (1.1b) to arrive at another basic equation:

(1.5) $$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Many other well-known results such as the law of sines and the law of cosines, as well as the sine and cosine of sums of angles can also be derived by similar trivial moves.

GENERALIZED SINE AND COSINE FUNCTIONS

When "triangle solving," it is really never necessary to consider angles greater than 90°. However, it is useful to generalize the sine and cosine functions to accommodate angles greater than this. The construction shown in Figure 1.2*a* helps to illustrate this generalization. Shown in the figure is a circle (of radius R) divided into four sections or quadrants. These quadrants are labeled from I to IV in the counterclockwise direction. The line drawn, at an angle θ, from the origin (center of circle) to the circumference is known as the radius vector. The angle θ is measured from the horizontal axis to the radius vector and is considered positive when swept in a counterclockwise direction.

We now construct the right triangle by dropping a line from the tip of the radius vector to the horizontal axis to form the opposite side (O) of the triangle. The value or "length" of this side is measured from the horizontal axis to the tip of the radius vector. A line drawn from the origin to the point of intersection of the O side and the horizontal axis gives the adjacent (A) side and completes the construction of the triangle. Again, the value or "length" of this side is measured from the origin and depending upon the angle θ will be positive or negative. The generalized sine and cosine functions of the angle θ are defined exactly as in Equations (1.1) only now we permit O and A to take on negative values. Note that the length of the radius vector is always considered positive. Depending upon the angle θ, we have four possible locations for the right triangle (in quadrants I through IV). For example, shown in Figure 1.2*b* is a value of θ that places the triangle in the second

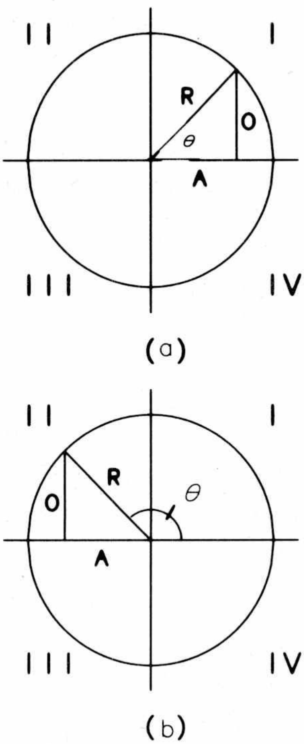

FIGURE 1.2. Generalized trigonometric relations.

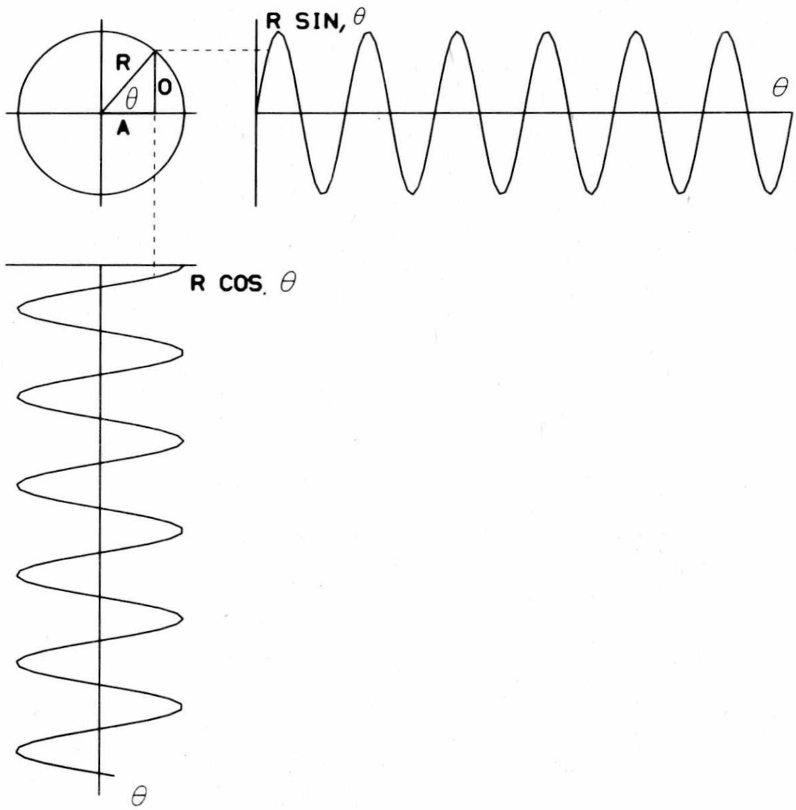

FIGURE 1.3. Periodic behavior of trigonometric functions.

quadrant. It is important to note that for these generalized sine and cosine functions the angle θ does not necessarily lie between the hypotenuse and the adjacent side of the triangle but is always measured from the positive horizontal axis to the radius vector in the counterclockwise direction.

The "triangle solving" equations still apply to the triangles regardless of the quadrant in which they are located. For example, in Figure 1.2*b* the angle between the hypotenuse and adjacent side is $180° - \theta$. Solving this triangle and comparing the results to the generalized trigonometric functions we obtain

$$\begin{aligned} \sin(180° - \theta) &= \sin\theta, \\ \cos(180° - \theta) &= -\cos\theta. \end{aligned} \tag{1.6}$$

Let us now carry this construction one step farther to illustrate the functional behavior of the sine and cosine.† In Figure 1.3 we again use the circle

† From here on we drop the term generalized sine and cosine functions and simply refer to them as the sine and cosine functions.

construction but now plot O ($R\sin\theta$) versus θ on the horizontal graph. This is a simple construction and is accomplished by graphically projecting the length of the side O over to the graph for various values of θ. This clearly illustrates the functional behavior of the sine function. A similar construction is shown for the cosine function in the vertical graph. Motion described by this type of function is called sinusoidal or simple harmonic motion. As is obvious from this figure, the sine and cosine functions repeat themselves every 2π radians or 360°, which corresponds to one complete revolution of the radius vector. Mathematically, we have:

$$\sin(\theta + 2\pi) = \sin\theta, \qquad \cos(\theta + 2\pi) = \cos\theta. \tag{1.7}$$

DEGREES, RADIANS, AND GRADS

We have been talking about degrees and radians as the measure of an angle rather casually so far. It is now time to be more specific. The common measure of the angle is the sexagesimal system in which the circle is divided into 360 basic units or degrees. The degree is divided into 60 minutes which, in turn, is divided into 60 seconds. A more reasonable measure of an angle is the grad. In this system the circle is divided into 400 grads or 100 grads per quadrant (this is an attempt at metrication of the circle). The rotational measure of an angle is also commonly used. If we consider one rotation of the vector in Figure 1.3 as our basic unit, then angles can be measured as fractions of a rotation. For example, $\frac{1}{5}$ rotation is equal to 72°.

Although the degree, grad, and rotation systems are sufficient to measure an angle for triangle solving, the radial measure is much more useful when discussing analytical properties of the trigonometric functions. As is well known, if we were to take the radius of a circle, bend it into an arc, and lay it along the circumference, it would take 6.28319..., or 2π, radial lengths to completely fit around the circle. Thus we can reasonably ask, "Suppose we take a fraction of the radius (bend it) and lay it on the circumference as shown in Figure 1.4; what angle would it span?"

We now have a measure of the angle θ in terms of the length of a fraction (possibly greater than one) of the radius $A'B'$. In fact, the angle θ is equal to the length of the arc AB for a unit circle. This is a most important concept and is very useful to us in the next section.

DERIVATIVES OF THE SINE AND COSINE FUNCTIONS

In this section we take a look at the derivatives of the sine and cosine functions. In line with our stated philosophy we use triangles or the classical

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