Michel Ledoux
Michel Talagrand

Ergebnisse der Mathematik und ihrer Grenzgebiete

3.Folge · Band 23

A Series of Modern Surveys in Mathematics

Probability in Banach Spaces

巴拿赫空间中的概率论

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To Marie-Françoise and WanSoo

Preface

This book tries to present some of the main aspects of the theory of Probability in Banach spaces, from the foundations of the topic to the latest developments and current research questions. The past twenty years saw intense activity in the study of classical Probability Theory on infinite dimensional spaces, vector valued random variables, boundedness and continuity of random processes, with a fruitful interaction with classical Banach spaces and their geometry. A large community of mathematicians, from classical probabilists to pure analysts and functional analysts, participated to this common achievement.

The recent use of isoperimetric tools and concentration of measure phenomena, and of abstract random process techniques has led today to rather a complete picture of the field. These developments prompted the authors to undertake the writing of this exposition based on this modern point of view.

This book does not pretend to cover all the aspects of the subject and of its connections with other fields. In spite of its ommissions, imperfections and errors, for which we would like to apologize, we hope that this work gives an attractive picture of the subject and will serve it appropriately.

In the process of this work, we benefited from the help of several people. We are grateful to A. de Acosta, K. Alexander, C. Borell, R. Dudley, X. Fernique, E. Giné, Y. Gordon, J. Kuelbs, W. Linde, M. B. Marcus, A. Pajor, V. Paulauskas, H. Queffélec, G. Schechtman, W. A. Woyczinski, M. Yor, J. Zinn for fruitful discussions (some of them over the years), suggestions, complements in references and their help in correcting mistakes and kind permission to include some results and ideas of their own. M. A. Arcones, E. Giné, J. Kuelbs and J. Zinn went in particular through the entire manuscript and we are mostly indebted to them for all their comments, remarks and corrections. Finally, special thanks are due to G. Pisier for his interest in this work and all his remarks. His vision has guided the authors over the years.

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Columbus, Paris, Strasbourg
January 1991

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Introduction

Probability in Banach spaces is a branch of modern mathematics that emphasizes the geometric and functional analytic aspects of Probability Theory. Its probabilistic sources may be found in the study of regularity of random processes (especially Gaussian processes) and Banach space valued random variables and their limiting properties, whose functional developments revealed and tied up strong and fruitful connections with classical Banach spaces and their geometry.

Probability in Banach spaces started perhaps in the early fifties with the study, by R. Fortet and E. Mourier, of the law of large numbers and the central limit theorem for sums of independent identically distributed Banach space valued random variables. Important contributions to the foundations of probability distributions on vector spaces, towards which A. N. Kolmogorov already pointed out in 1935, were at the time those of L. Le Cam and Y. V. Prokhorov and the Russian school. A decisive step to the modern developments of Probability in Banach spaces was the introduction by A. Beck (1962) of a convexity condition on normed linear spaces equivalent to the validity of the extension of a classical law of large numbers of Kolmogorov. This geometric line of investigation was pursued and amplified by the Schwartz school in the early seventies. The concept of radonifying and summing operators and the landmark work of B. Maurey and G. Pisier on type and cotype of Banach spaces considerably influenced the developments of Probability in Banach spaces. Other noteworthy achievements of the period are the early book (1968) by J.-P. Kahane, who systematically developed the crucial idea of symmetrization, and the study by J. Hoffmann-Jørgensen of sums of independent vector valued random variables. Simultaneously, the study of regularity of random processes, in particular Gaussian processes, saw great progress in the late sixties and early seventies with the introduction of entropy methods. Processes are understood here as random functions on some abstract index set T, in other words as families $X = (X_t)_{t \in T}$ of random variables. In this setting of what might appear as Probability with minimal structure, the major discovery of V. Strassen, V. N. Sudakov and R. Dudley (1967) was the idea of analyzing regularity properties of a Gaussian process X through the geometry of the index set T for the L_2-metric $\|X_s - X_t\|_2$ induced by X itself. These foundations of Probability in Banach spaces led to a rather intense activity for the last fifteen years. In particular, the Dudley-Fernique theorems on boundedness and continuity of Gaussian and stationary Gaussian processes allowed the definitive treatment by M. B. Marcus and G. Pisier of regularity of random Fourier series, initiated in this line by J.-P. Kahane. With the concepts of type and cotype, limit theorems for sums of independent Banach space valued random variables were appropriately described. Under the impulse, in particular, of the local theory of Banach spaces, isoperimetric methods and concentration of measure phenomena, put forward most vigorously by V. D. Milman, made a strong entry in the subject during the late seventies and eighties. Starting from Dvoretzky's theorem on almost Euclidean sections of convex bodies, the isoperimetric inequalities on spheres and in Gauss space proved most powerful in the study of Gaussian measures and processes, in particular through the work by C. Borell. They were useful too in the study of limit theorems through the technique of randomization. An important recent development was the discovery, motivated by these results, of a new isoperimetric inequality for subsets of a product of probability spaces that is closely connected to the tail behavior of sums of independent Banach space valued random variables. It gives in particular today an almost complete description of various strong limit theorems like the classical laws of large numbers and the law of the iterated logarithm. In the mean time, almost sure boundedness and continuity of general Gaussian processes have been completely understood with the tool of majorizing measures.

One of the fascinations of the theory of Probability in Banach spaces today is its use of a wide range of rather powerful methods. Since the field is one of the most active contact points between Probability and Analysis, it should be no surprise that many of the techniques are not probabilistic but rather come from Analysis. The book focuses on two connected topics – the use of isoperimetric methods and regularity of random processes – where many of these techniques come into play and which encompass many (although not all) of the main aspects of Probability in Banach spaces. The purpose of this book is to give a modern and, at many places, seemingly definitive account on these topics, from the foundations of the theory to the latest research questions. The book is written so as to require only basic prior knowledge of either Probability or Banach space theory, in order to make it accessible from readers of both fields as well as to non-specialists. It is moreover presented in perspective with the historical developments and strong modern interactions between Measure and Probability theory, Functional Analysis and Geometry of Banach spaces. It is essentially self-contained (with the exception that the proof of a few deep isoperimetric results have not been reproduced), so as to be accessible to anyone starting the subject, including graduate students. Emphasis has been put in bringing forward the ideas we judge important but not on encyclopedic detail. We hope that these ideas will fruitfully serve the further developments of the field and hope that their propagation will influence other or new areas.

This book emphasizes the recent use of isoperimetric inequalities and related concentration of measure phenomena, and of modern random process techniques in Probability in Banach spaces. The two parts are introduced by chapters on isoperimetric background and generalities on vector valued random variables. To explain and motivate the organization of our work, let us briefly analyze one fundamental example. Let (T,d) be a compact metric space and let $X = (X_t)_{t\in T}$ be a Gaussian process indexed by T. If X has almost all its sample paths continuous on (T,d), it defines a Gaussian Radon probability measure on the Banach space $C(T)$ of all continuous functions on T. Such a Gaussian measure or variable may then be studied for its own sake and shares indeed some remarkable integrability and tail behavior properties of isoperimetric nature. On the other hand, one might wonder (before) when a given Gaussian process is almost surely continuous. As we have seen, an analysis of the geometry of the index set T for the L_2-metric $\|X_s - X_t\|_2$ induced by the process yields a complete understanding of this property. These related but somewhat different aspects of the study of Gaussian variables, which were historically the two main streams of developments, led us thus to divide the book into two parts. (The logical order would have been perhaps to ask first when a given process is bounded or continuous and then investigate it for its properties as a well defined infinite dimensional random vector; we have however chosen the other way for various pedagogical reasons.) In the first part, we study vector valued random variables, their integrability and tail behavior properties and strong limit theorems for sums of independent random variables. Successively, vector valued Gaussian variables, Rademacher series, stable variables and sums of independent random variables are investigated in this scope using recent isoperimetric tools. The strong law of large numbers and the law of the iterated logarithm, for which the almost sure statement is shown to reduce to the statement in probability, complete this first part with extensions to infinite dimensional Banach space valued random variables of some classical real limit theorems. In the second part, tightness of sums of independent random variables and regularity properties of random processes are presented. The link with the Geometry of Banach spaces through type and cotype is developed with applications in particular to the central limit theorem. General random processes are then investigated and regularity of Gaussian processes characterized via majorizing measures with applications to random Fourier series. The book is completed with an account on empirical process methods and with several applications, especially to local theory of Banach spaces, of the probabilistic ideas presented in this work. A diagram describes some of the interactions between the two main parts of the book and the natural connections between the various chapters.

We would like to mention that the topics of Probability in Banach spaces selected in this book are not exhaustive and actually only reflect the tastes and interests of the authors. Among the topics not covered, let us mention especially martingales with values in Banach spaces and their relations to Geometry. We refer to [D-U] (on the Radon-Nikodym Property), [Schw3], [Pi16] (on convexity and smoothness) and [Bu] (on Unconditional Martingale Differences and ζ-convexity) for an account on this deep and fruitful subject as well as for detailed references for further reading. Empirical processes are only briefly treated in this book and the interested reader will find in [Du5], [Ga],