Classification of Pseudo-reductive Groups

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Introduction

1.1 Motivation

Algebraic and arithmetic geometry in positive characteristic provide important examples of imperfect fields, such as (i) Laurent-series fields over finite fields and (ii) function fields of positive-dimensional varieties (even over an algebraically closed field of constants). Generic fibers of positive-dimensional algebraic families naturally lie over a ground field as in (ii).

For a smooth connected affine group G over a field k, the unipotent radical $\mathscr{R}_u(G_{\overline{k}}) \subset G_{\overline{k}}$ may not arise from a k-subgroup of G when k is imperfect. (Examples of this phenomenon will be given shortly.) Thus, for the maximal smooth connected unipotent normal k-subgroup $\mathscr{R}_{u,k}(G) \subset G$ (the *k-unipotent radical*), the quotient $G/\mathscr{R}_{u,k}(G)$ may not be reductive when k is imperfect.

A *pseudo-reductive group* over a field k is a smooth connected affine k-group G such that $\mathscr{R}_{u,k}(G)$ is trivial. For any smooth connected affine k-group G, the quotient $G/\mathscr{R}_{u,k}(G)$ is pseudo-reductive. A pseudo-reductive k-group G that is perfect (i.e., G equals its derived group $\mathscr{D}(G)$) is called *pseudo-semisimple*. If k is perfect then pseudo-reductive k-groups are connected reductive k-groups by another name. For imperfect k the situation is completely different:

Example 1.1.1. Weil restrictions $G = \mathrm{R}_{k'/k}(G')$ for finite extensions k'/k and connected reductive k'-groups G' are pseudo-reductive [CGP, Prop. 1.1.10]. If G' is nontrivial and k'/k is not separable then such G are *never* reductive [CGP, Ex. 1.6.1]. A solvable pseudo-reductive group is necessarily commutative [CGP, Prop. 1.2.3], but the structure of commutative pseudo-reductive groups appears to be intractable (see [T]). The quotient of a pseudo-reductive k-group by a smooth connected normal k-subgroup or by a central closed k-subgroup scheme can fail to be pseudo-reductive, and a smooth connected normal k-subgroup of a pseudo-semisimple k-group can fail to be perfect; see [CGP, Ex. 1.3.5, 1.6.4] for such examples over any imperfect field k.

A typical situation where the structure theory of pseudo-reductive groups is useful is in the study of smooth affine k-groups about which one has limited information but for which one wishes to prove a general theorem (e.g., cohomological finiteness); examples include the Zariski closure in GL_n of a subgroup of $\mathrm{GL}_n(k)$, and the maximal smooth k-subgroup of a schematic stabilizer (as in local-global problems). For questions not amenable to study over $\overline{k}$ when k is imperfect, this structure theory makes possible what had previously seemed out of reach over such k: to reduce problems for general smooth affine k-groups to the reductive and commutative cases (over finite extensions of k). Such procedures are essential to prove finiteness results for degree-1 Tate-Shafarevich sets of arbitrary affine group schemes of finite type over global function fields, even in the general *smooth* affine case; see [C1, §1] for this and other applications.

A detailed study of pseudo-reductive groups was initiated by Tits; he constructed several instructive examples and his ultimate goal was a classification. The general theory developed in [CGP] by characteristic-free methods includes the open cell, root systems, rational conjugacy theorems, the Bruhat decomposition for rational points, and a structure theory "modulo the commutative case" (summarized in [C1, §2] and [R]). The lack of a concrete description of commutative pseudo-reductive groups is not an obstacle in applications (see [C1]).

In general, if G is a smooth connected affine k-group then $\mathscr{R}_{u,k}(G)_K \subset \mathscr{R}_{u,K}(G_K)$ for any extension field K/k, and this inclusion is an equality when K is separable over k [CGP, Prop. 1.1.9] but generally not otherwise (e.g., equality fails with $K = \overline{k}$ for any imperfect k and non-reductive pseudo-reductive G). Taking $K = k_s$ shows that G is pseudo-reductive if and only if G_{k_s} is pseudo-reductive (and also shows that if k is perfect then pseudo-reductive k-groups are precisely connected reductive k-groups). Hence, any smooth connected normal k-subgroup of a pseudo-reductive k-group is pseudo-reductive.

Every smooth connected affine k-group G is generated by $\mathscr{D}(G)$ and a single Cartan k-subgroup. Since $\mathscr{D}(G)$ is pseudo-semisimple when G is pseudo-reductive [CGP, Prop. 1.2.6], and Cartan k-subgroups of pseudo-reductive k-groups are commutative and pseudo-reductive, the main work in describing pseudo-reductive groups lies in the pseudo-semisimple case. A smooth affine k-group G is *pseudo-simple* (over k) if it is pseudo-semisimple, nontrivial, and has no nontrivial smooth connected proper normal k-subgroup; it is *absolutely pseudo-simple* if G_{k_s} is pseudo-simple. (See [CGP, Def. 3.1.1, Lemma 3.1.2] for equivalent formulations.) A pseudo-reductive k-group G is *pseudo-split* if it contains a split maximal k-torus T, in which case any two such tori are conjugate by an element of $G(k)$ [CGP, Thm. C.2.3]

Remark 1.1.2. If G is a pseudo-semisimple k-group then the set $\{G_i\}$ of its pseudo-simple normal k-subgroups is finite, the G_i's pairwise commute and generate G, and every perfect smooth connected normal k-subgroup of G is generated by the G_i's that it contains (see [CGP, Prop. 3.1.8]). The core of the study of pseudo-reductive groups G is the absolutely pseudo-simple case.

Although [CGP] gives general structural foundations for the study and application of pseudo-reductive groups over any imperfect field k, there are natural topics not addressed in [CGP] whose development requires new ideas, such as:

(i) Are there versions of the Isomorphism and Isogeny Theorems for pseudo-split pseudo-reductive groups and of the Existence Theorem for pseudo-split pseudo-simple groups?
(ii) The *standard construction* (see §2.1) is exhaustive when $p := \text{char}(k) \neq 2, 3$. Incorporating constructions resting on exceptional isogenies [CGP, Ch. 7–8] and birational group laws [CGP, §9.6–§9.8] gives an analogous result when $p = 2, 3$ provided that $[k : k^2] = 2$ if $p = 2$; see [CGP, Thm. 10.2.1, Prop. 10.1.4]. More examples exist if $p = 2$ and $[k : k^2] > 2$ (see §1.3); can we generalize the standard construction for such k?
(iii) Is the automorphism functor of a pseudo-semisimple group representable? (Representability fails in the commutative pseudo-reductive case.) If so, what can be said about the structure of the identity component and component group of its maximal smooth closed subgroup $\text{Aut}^{\text{sm}}_{G/k}$ (thereby defining a notion of "pseudo-inner" k_s/k-form via $(\text{Aut}^{\text{sm}}_{G/k})^0$)?
(iv) What can be said about existence and uniqueness of pseudo-split k_s/k-forms, and of quasi-split pseudo-inner k_s/k-forms? ("Quasi-split" means the existence of a solvable pseudo-parabolic k-subgroup.)
(v) Is there a Tits-style classification in the pseudo-semisimple case recovering the version due to Tits in the semisimple case? (Many ingredients in the semisimple case *break down* for pseudo-semisimple G; e.g., G may have no pseudo-split k_s/k-form, and the quotient G/Z_G of G modulo the scheme-theoretic center Z_G can be a proper k-subgroup of $(\text{Aut}^{\text{sm}}_{G/k})^0$.)

The special challenges of characteristic 2 are reviewed in §1.3–§1.4 and §4.2. Recent work of Gabber on compactification theorems for arbitrary linear algebraic groups uses the structure theory of pseudo-reductive groups over general (imperfect) fields. That work encounters additional complications in characteristic 2 which are overcome via the description of pseudo-reductive groups as central extensions of groups obtained by the "generalized standard" construction given in Chapter 9 of this monograph (see the Structure Theorem in §1.6).

1.2 Root systems and new results

A maximal k-torus T in a pseudo-reductive k-group G is an almost direct product of the maximal central k-torus Z in G and the maximal k-torus $T' := T \cap \mathscr{D}(G)$ in $\mathscr{D}(G)$ [CGP, Lemma 1.2.5]. Suppose T is *split*, so the set $\Phi := \Phi(G, T)$ of nontrivial T-weights on $\mathrm{Lie}(G)$ injects into $\mathrm{X}(T')$ via restriction.

The pair $(\Phi, \mathrm{X}(T')_{\mathbf{Q}})$ is always a root system (coinciding with $\Phi(\mathscr{D}(G), T')$ since $G/\mathscr{D}(G)$ is commutative) [CGP, Thm. 2.3.10], and can be canonically enhanced to a root datum [CGP, §3.2]. In particular, to every pseudo-semisimple k_s-group we may attach a *Dynkin diagram*. However, $(\Phi, \mathrm{X}(T')_{\mathbf{Q}})$ can be non-reduced when k is imperfect of characteristic 2 (the non-multipliable roots are the roots of the maximal geometric reductive quotient $G_{\overline{k}}^{\mathrm{red}}$). A pseudo-split pseudo-semisimple group is (absolutely) pseudo-simple precisely when its root system is irreducible [CGP, Prop. 3.1.6].

This monograph builds on earlier work [CGP] via new techniques and constructions to answer the questions (i)–(v) raised in §1.1. In so doing, we also simplify the proofs of some results in [CGP]. (For instance, the standardness of all pseudo-reductive k-groups if $\mathrm{char}(k) \neq 2, 3$ is recovered here by another method in Theorem 3.4.2.) Among the new results in this monograph are:

- (i) pseudo-reductive versions of the Existence, Isomorphism, and Isogeny Theorems (see Theorems 3.4.1, 6.1.1, and A.1.2),
- (ii) a *structure theorem* over arbitrary imperfect fields k (see §1.5–§1.6),
- (iii) existence of the automorphism scheme $\mathrm{Aut}_{G/k}$ for pseudo-semisimple G, and properties of the identity component and component group of its maximal smooth closed k-subgroup $\mathrm{Aut}_{G/k}^{\mathrm{sm}}$ (see Chapter 6),
- (iv) uniqueness and optimal existence results for pseudo-split and "quasi-split" k_s/k-forms for imperfect k, including examples (in *every* positive characteristic) where existence *fails* (see §1.7),
- (v) a Tits-style classification of pseudo-semisimple k-groups G in terms of both the Dynkin diagram of G_{k_s} with $*$-action of $\mathrm{Gal}(k_s/k)$ on it and the k-isomorphism class of the embedded anisotropic kernel (see §1.7).

We illustrate (v) in Appendix D by using anisotropic quadratic forms over k to construct and classify absolutely pseudo-simple groups of type F_4 with k-rank 2 (which never exist in the semisimple case).

1.3 Exotic groups and degenerate quadratic forms

If $p = 2$ and $[k : k^2] > 2$ then there exist families of *non-standard* absolutely pseudo-simple k-groups of types B_n, C_n, and BC_n (for every $n \geqslant 1$) with no analogue when $[k : k^2] = 2$. Their existence is explained by a construction with certain *degenerate* quadratic spaces over k that exist only if $[k : k^2] > 2$:

Example 1.3.1. Let (V, q) be a quadratic space over a field k with $\mathrm{char}(k) = 2$, $d := \dim V \geqslant 3$, and $q \neq 0$. Let $B_q : (v, w) \mapsto q(v + w) - q(v) - q(w)$ be the associated symmetric bilinear form and $V^{\perp}$ the defect space consisting of $v \in V$ such that the linear form $B_q(v, \cdot)$ on V vanishes. The restriction $q|_{V^{\perp}}$ is 2-linear (i.e., additive and $q(cv) = c^2 q(v)$ for $v \in V$, $c \in k$) and $\dim(V/V^{\perp}) = 2n$ for some $n \geqslant 0$ since B_q induces a non-degenerate symplectic form on $V/V^{\perp}$.

Assume $0 < \dim V^{\perp} < \dim V$. Now q is non-degenerate (i.e., the projective hypersurface $(q = 0) \subset \mathbf{P}(V^*)$ is k-smooth) if and only if $\dim V^{\perp} = 1$, which is to say $d = 2n + 1$. It is well-known that in such cases $\mathrm{SO}(q)$ is an absolutely simple group of type B_n with $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$, so $\mathrm{SO}(q)$ is the maximal smooth closed k-subgroup of $\mathrm{O}(q)$ since $\mathrm{char}(k) = 2$. Assume also that (V, q) is *regular*; i.e., $\ker(q|_{V^{\perp}}) = 0$. Regularity is preserved by any separable extension on k (Lemma 7.1.1). For such (possibly degenerate) q, define $\mathrm{SO}(q)$ to be the maximal smooth closed k-subgroup of the k-group scheme $\mathrm{O}(q)$; i.e., $\mathrm{SO}(q)$ is the k-descent of the Zariski closure of $\mathrm{O}(q)(k_s)$ in $\mathrm{O}(q)_{k_s}$. In §7.1–§7.3 we prove: $\mathrm{SO}(q)$ is absolutely pseudo-simple with root system B_n over k_s where $2n = \dim(V/V^{\perp})$, the dimension of a root group of $\mathrm{SO}(q)_{k_s}$ is 1 for long roots and $\dim V^{\perp}$ for short roots, and the minimal field of definition over k for the geometric unipotent radical of $\mathrm{SO}(q)$ is the k-finite subextension $K \subset k^{1/2}$ generated over k by the square roots $(q(v')/q(v))^{1/2}$ for nonzero $v, v' \in V^{\perp}$.

For any nonzero $v_0 \in V^{\perp}$, the map $v \mapsto (q(v)/q(v_0))^{1/2}$ is a k-linear injection of $V^{\perp}$ into $k^{1/2}$ with image $\mathscr{V}$ containing 1 and generating K as a k-algebra. If we replace v_0 with a nonzero $v_1 \in V^{\perp}$ then the associated k-subspace of K is $\lambda\mathscr{V}$ where $\lambda = (q(v_0)/q(v_1))^{1/2} \in K^{\times}$. In particular, the case $K \neq k$ occurs if and only if $\dim V^{\perp} \geqslant 2$, which is precisely when the regular q is *degenerate*, and always $[k : k^2] = [k^{1/2} : k] \geqslant \dim V^{\perp}$. If $V^{\perp} = K$, as happens whenever $[k : k^2] = 2$, then $\mathrm{SO}(q)$ is the quotient of a "basic exotic" k-group [CGP, §7.2] modulo its center. The $\mathrm{SO}(q)$'s with $V^{\perp} \neq K$ (so $[k : k^2] > 2$) are a new class of absolutely pseudo-simple k-groups of type B_n (with trivial center); for $n = 1$ and isotropic q these are the type-A_1 groups $\mathrm{P}H_{V^{\perp}, K/k}$ built in §3.1.

In §7.2–§7.3 we show that every k-isomorphism $\mathrm{SO}(q') \simeq \mathrm{SO}(q)$ arises from a conformal isometry $q' \simeq q$ and use this to construct more absolutely pseudo-simple k-groups of type B with trivial center via geometrically integral non-smooth quadrics in Severi–Brauer varieties associated to certain elements of order 2 in the Brauer group $\mathrm{Br}(k)$. Remarkably, this accounts for *all* non-reductive pseudo-reductive groups whose Cartan subgroups are tori (see Proposition 7.3.7), and when combined with the exceptional isogeny $\mathrm{Sp}_{2n} \to \mathrm{SO}_{2n+1}$ in characteristic 2 via a fiber product construction it yields (in §8.2) new absolutely pseudo-simple groups of type C_n when $n \geqslant 2$ and $[k : k^2] > 2$ (with short root groups over k_s of dimension $[K : k]$ and long root groups over k_s of dimension $\dim V^{\perp}$). A generalization in §8.3 gives *even more* such k-groups for $n = 2$ if $[k : k^2] > 8$ (using that $\mathrm{B}_2 = \mathrm{C}_2$). In §1.5–§1.6 we provide a context for this zoo of constructions.

1.4 Tame central extensions

A new ingredient in this monograph is a generalization of the "standard construction" (from §2.1) that is better-suited to the peculiar demands of characteristic 2. Before we address that, it is instructive to recall the principle underlying the ubiquity of standardness *away from* the case $\mathrm{char}(k) = 2$ with $[k : k^2] > 2$, via splitting results for certain classes of central extensions. We now review the most basic instance of such splitting, to see why it breaks down completely (and hence new methods are required) when $\mathrm{char}(k) = 2$ with $[k : k^2] > 2$ (see 1.4.2).

1.4.1. Let G be an absolutely pseudo-simple k-group with minimal field of definition K/k for $\mathscr{R}_u(G_{\overline{k}}) \subset G_{\overline{k}}$, and let $G' := G_K^{\mathrm{ss}}$ be the maximal semisimple quotient of G_K. For the simply connected central cover $q : \widetilde{G}' \to G'$ and $\mu := \ker q \subset Z_{\widetilde{G}'}$, there is (as in [CGP, Def. 5.3.5]) a canonical k-homomorphism

$$\xi_G : G \to \mathscr{D}(\mathrm{R}_{K/k}(G')) = \mathrm{R}_{K/k}(\widetilde{G}')/\mathrm{R}_{K/k}(\mu) \tag{1.4.1.1}$$

induced by the natural map $i_G : G \to \mathrm{R}_{K/k}(G')$. The map ξ_G makes sense for any pseudo-reductive G but (as in [CGP]) it is of interest only for absolutely pseudo-simple G. By Proposition 2.3.4, $\ker \xi_G$ is central if $\mathrm{char}(k) \neq 2$.

The key to the proof that G is standard if $\mathrm{char}(k) \neq 2, 3$ is the surjectivity of ξ_G for such k, as then (1.4.1.1) pulls back to a central extension E of $\mathrm{R}_{K/k}(\widetilde{G}')$ by $\ker \xi_G$. This central extension is *split* due to a general fact: if k'/k is an arbitrary finite extension of fields and $\mathscr{G}'$ a connected semisimple k'-group that is *simply connected* then for any commutative affine k-group scheme Z of finite type with no nontrivial smooth connected k-subgroup (e.g., $Z = \ker \xi_G$ as