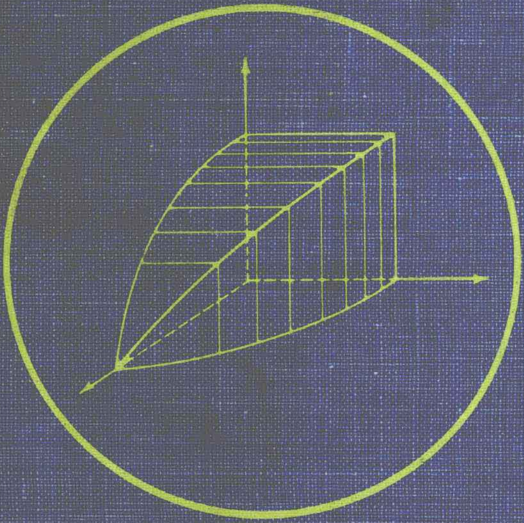
# A SECOND COURSE IN CALCULUS



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# A Second Course in Calculus

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# A Second Course in Calculus

# Preface

This text, designed for a second year calculus course, can follow any standard first year course in one-variable calculus. Its purpose is to cover the material most useful at this level, to maintain a balance between theory and practice, and to develop techniques and problem solving skills.

The topics fall into several categories:

*Infinite series and integrals*

Chapter 1 covers convergence and divergence of series and integrals. It contains proofs of basic convergence tests, relations between series and integrals, and manipulation with geometric, exponential, and related series. Chapter 2 covers approximation of functions by Taylor polynomials, with emphasis on numerical approximations and estimates of remainders. Chapter 3 deals with power series, including intervals of convergence, expansions of functions, and uniform convergence. It features calculations with series by algebraic operations, substitution, and term-by-term differentiation and integration.

*Vector methods*

Vector algebra is introduced in Chapter 4 and applied to solid analytic geometry. The calculus of one-variable vector functions and its applications to space curves and particle mechanics comprise Chapter 5.

*Linear algebra*

Chapter 7 contains a practical introduction to linear algebra in two and three dimensions. We do not attempt a complete treatment of foundations, but rather limit ourselves to those topics that have immediate application to calculus. The main topics are linear transformations in $\mathbf{R}^2$ and $\mathbf{R}^3$, their matrix representations, manipulation with matrices, linear systems, quadratic forms, and quadric surfaces.

*Differential calculus of several variables*

Chapter 6 contains preliminary material on sets in the plane and space, and the definition and basic properties of continuous functions. This is followed by partial derivatives with applications to maxima and minima. Chapter 8 continues with a careful treatment of differentiability and applications to tangent planes, gradients, directional derivatives, and differentials. Here ideas from linear algebra are used judiciously. Chapter 9 covers higher order partial derivatives, Taylor polynomials, and second derivative tests for extrema.

*Multiple integrals*

In Chapters 10 and 11 we treat double and triple integrals intuitively, with emphasis on iteration, geometric and physical applications, and coordinate changes. In Chapter 12 we develop the theory of the Riemann integral starting with step functions. We continue with Jacobians and the change of variable formula, surface area, and Green's Theorem.

*Differential equations*

Chapter 13 contains an elementary treatment of first order equations, with emphasis on linear equations, approximate solutions, and applications. Chapter 14 covers second order linear equations and first order linear systems, including matrix series solutions. These chapters can be taken up any time after Chapter 7.

*Complex analysis*

The final chapter moves quickly through basic complex algebra to complex power series, shortcuts using the complex exponential function, and applications to integration and differential equations.

## Features

The key points of one-variable calculus are reviewed briefly as needed.

Optional topics are scattered throughout, for example Stirling's Formula, characteristic roots and vectors, Lagrange multipliers, and Simpson's Rule for double integrals.

Numerous worked examples teach practical skills and demonstrate the utility of the theory.

We emphasize simple line drawings that a student can learn to do himself.

## Acknowledgments

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# Contents

# 1. Infinite Series and Integrals

## 1. INFINITE SERIES

One of the most important topics in mathematical analysis, both in theory and applications, is infinite series. The basic problem is how to add up a sum with infinitely many terms. At first that seems impossible; life is too short. However, suppose we look at the sum

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots$$

and start adding up terms. We find 1, $\frac{3}{2}$, $\frac{7}{4}$, $\frac{15}{8}$, $\frac{31}{16}$, $\cdots$, numbers getting closer and closer to 2. The message is clear: in some limit sense the total of all the terms is 2.

If we try to add up terms of the sum

$$1 + 1 + 1 + \cdots,$$

we find 1, 2, 3, 4, $\cdots$, numbers becoming larger and larger. The situation is hopeless; there is no reasonable total.

Let us now consider in some detail two important infinite sums.

### *Geometric Series*

A **geometric series** is an infinite sum in which the ratio of any two consecutive terms is always the same:

$$a + ar + ar^2 + \cdots + ar^n + \cdots \qquad (a \neq 0, \quad r \neq 0).$$

Let $s_n$ denote the sum of all terms up to $ar^n$,

$$s_n = a + ar + ar^2 + \cdots + ar^n.$$

If $r = 1$, then $s_n = a + a + \cdots + a = (n + 1)a$, so $s_n \longrightarrow \pm\infty$. If $r \neq 1$, there is a simple formula for $s_n$:

$$s_n = a(1 + r + r^2 + \cdots + r^n) = a\left(\frac{1 - r^{n+1}}{1 - r}\right).$$

(To check, multiply both sides by $1 - r$.) If $|r| < 1$, then $r^{n+1} \longrightarrow 0$ as $n$ increases. Hence a logical choice for the "sum" of the geometric series is

$a/(1-r)$. But if $|r| > 1$, then $r^{n+1}$ grows beyond all bound, and the situation is hopeless. If $r = -1$, then $s_n$ is alternately $a$ and 0. There is no reasonable sum in this case either.

> An infinite geometric series
>
> $$a + ar + ar^2 + \cdots + ar^n + \cdots$$
>
> has the sum $a/(1-r)$ if $|r| < 1$, but no sum if $|r| \geq 1$.

### *Harmonic Series*

The series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is known as the **harmonic series.** It is not at all obvious, but the sums $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + n^{-1}$ increase beyond all bound, so the series has no sum. To see why, we observe that

$$s_1 = 1 > \frac{1}{2},$$

$$s_2 = s_1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = \frac{2}{2},$$

$$s_4 = s_2 + \left(\frac{1}{3} + \frac{1}{4}\right) > s_2 + \left(\frac{1}{4} + \frac{1}{4}\right) > \frac{2}{2} + \frac{1}{2} = \frac{3}{2},$$

$$s_8 = s_4 + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > s_4 + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) > \frac{3}{2} + \frac{1}{2} = \frac{4}{2}.$$

Similarly, $s_{16} > 5/2$, $s_{32} > 6/2$, $\cdots$, $s_{2^n} > (n+1)/2$. Now the sequence of sums $s_n$ increases, and our estimates show $s_n$ eventually passes any given positive number. (This happens very slowly it is true; around $2^{15}$ terms are needed before $s_n$ exceeds 10 and around $2^{29}$ terms before it exceeds 20.)

REMARK: Both the geometric series for $0 < r < 1$ and the harmonic series have positive terms that decrease toward zero, yet one series has a sum and the other does not. This indicates the subtlety we must expect in our further study of infinite series.

## EXERCISES

Find the sum:

1. $1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^9}$
2. $1 - \frac{1}{3} + \frac{1}{3^2} - + \cdots - \frac{1}{3^9}$

3. $\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{256}$

4. $\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots+\left(\frac{2}{3}\right)^6$

5. $3+\frac{3^2}{x}+\frac{3^3}{x^2}+\cdots+\frac{3^{n+1}}{x^n}$

6. $1-y^2+y^4-+\cdots+y^{20}$

7. $r^{1/2}+r+r^{3/2}+\cdots+r^4$

8. $(x+1)+(x+1)^2+\cdots+(x+1)^5$.

Find the sum of the series:

9. $1-\frac{2}{5}+\left(\frac{2}{5}\right)^2-\left(\frac{3}{5}\right)^3+-\cdots$

10. $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+-\cdots$

11. $\frac{1}{2^{10}}+\frac{1}{2^{11}}+\frac{1}{2^{12}}+\cdots$

12. $\frac{1}{3}+\frac{1}{27}+\frac{1}{243}+\cdots$

13. $\frac{1}{2+x^2}+\frac{1}{(2+x^2)^2}+\frac{1}{(2+x^2)^3}+\cdots$

14. $\frac{\cos\theta}{2}+\frac{\cos^2\theta}{4}+\frac{\cos^3\theta}{8}+\cdots$.

15. A certain rubber ball when dropped will bounce back to half the height from which it is released. If the ball is dropped from 3 ft and continues to bounce indefinitely, find the total distance through which it moves.

16. Trains $A$ and $B$ are 60 miles apart on the same track and start moving toward each other at the rate of 30 mph. At the same time, a fly starts at train $A$ and flies to train $B$ at 60 mph. Then it returns to train $A$, then to $B$, etc. Use a geometric series to compute the total distance it flies until the trains meet.

17. (cont.) Do Ex. 16 without geometric series.

18. A line segment of length $L$ is drawn and its middle third is erased. Then (step 2) the middle third of each of the two remaining segments is erased. Then (step 3) the middle third of each of the four remaining segments is erased, etc. After step $n$, what is the total length of all the segments deleted?

Interpret the repeating decimals as geometric series and find their sums:

19. $0.11111\cdots$

20. $0.101010\cdots$

21. $0.434343\cdots$

22. $0.185185185\cdots$.

Show that the series have no sums:

23. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots$

24. $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots$.

25. Find $n$ so large that

$$\frac{1}{101}+\frac{1}{102}+\cdots+\frac{1}{n}>2.$$

26. Aristotle summarized Zeno's paradoxes as follows:

> I can't go from here to the wall. For to do so, I must first cover half the distance, then half the remaining distance, then again half of what still remains. This process can always be continued and can never be completed.

Explain what is going on here.

## 2. CONVERGENCE AND DIVERGENCE

It is time to formulate the ideas of Section 1 more precisely.

> An **infinite series** is a formal sum
>
> $$a_1 + a_2 + a_3 + \cdots.$$
>
> Associated with each infinite series is its sequence $\{s_n\}$ of **partial sums** defined by
>
> $$s_1 = a_1, \qquad s_2 = a_1 + a_2, \qquad \cdots, \qquad s_n = a_1 + a_2 + \cdots + a_n.$$
>
> A series **converges** to the number $S$, or has **sum** $S$, if $\lim_{n\to\infty} s_n = S$. A series **diverges**, or has no sum, if $\lim_{n\to\infty} s_n$ does not exist.

A series that converges is called **convergent**; a series that diverges is called **divergent**.

Let us recall the meaning of the statement $\lim_{n\to\infty} s_n = S$. Intuitively, it means that as $N$ grows larger and larger, the greatest distance $|s_n - S|$, for *all* $n \geq N$, becomes smaller and smaller. Precisely, for each $\epsilon > 0$, there is a positive integer $N$ such that $|s_n - S| < \epsilon$ for all $n \geq N$. Let us rephrase the definition of convergence accordingly.

> The infinite series $a_1 + a_2 + a_3 + \cdots$ converges to $S$ if for each $\epsilon > 0$, there is a positive integer $N$ such that
>
> $$|(a_1 + a_2 + \cdots + a_n) - S| < \epsilon$$
>
> whenever $n \geq N$.

Thus, no matter how small $\epsilon$, you will get within $\epsilon$ of $S$ by adding up enough terms. For each $\epsilon$, the $N$ tells how many terms are "enough". Naturally the smaller $\epsilon$ is, the larger $N$ will be. From the way convergence is defined, the study of infinite series is really the study of *sequences* of partial sums. Hence we may apply everything we know about sequences.

We know that inserting, deleting, or altering any finite number of elements of a sequence does not affect its convergence or divergence. The same holds for series. For instance, if we delete the first 10 terms of the series $a_1 + a_2 + a_3 + \cdots$, then we decrease each partial sum $s_n$ (for $n > 10$) by the amount $a_1 + a_2 + \cdots + a_{10}$. If the original series diverges, then so does the modified series. If it converges to $S$, then the modified series converges to $S - (a_1 + a_2 + \cdots + a_{10})$.

Warning: In problems where we must decide whether a given infinite series converges or diverges, we shall often, without prior notice, ignore or change a (finite) batch of terms at the beginning. This, we now know, does not affect convergence.

### Notation

The first term of a series need not be $a_1$. Often it is convenient to start with $a_0$ or with some other $a_k$.

It is also convenient to use summation notation and abbreviate $a_1 + a_2 + a_3 + \cdots$ by $\sum_{n=1}^{\infty} a_n$, and even simply $\sum a_n$. In summation notation, the partial sums $s_n$ of an infinite series $\sum_{n=1}^{\infty} a_n$ are given by

$$s_n = \sum_{k=1}^{n} a_k.$$

### Cauchy Criterion

Recall the Cauchy criterion for convergence of sequences:

A sequence $\{s_n\}$ converges if and only if for each $\epsilon > 0$, there is a positive integer $N$ such that

$$|s_m - s_n| < \epsilon$$

whenever $m, n \geq N$.

Thus all elements of the sequence beyond a certain point must be within $\epsilon$ of each other. The advantage of the Cauchy criterion is that it depends only on the elements of the sequence itself; you don't have to know the limit of a sequence in order to show convergence. That's a great help; sometimes it is very hard to find the exact limit of a sequence, whereas you may only need to know that the sequence does indeed converge to some limit.

Let us apply the Cauchy criterion to the partial sums of a series. We simply observe (for $m > n$) that

$$\begin{aligned} s_m - s_n &= (a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots + a_m) - (a_1 + a_2 + \cdots + a_n) \\ &= a_{n+1} + a_{n+2} + \cdots + a_m. \end{aligned}$$

**Cauchy Test** An infinite series $\sum a_n$ converges if and only if for each $\epsilon > 0$, there is a positive integer $N$ such that

$$|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon$$

whenever $m > n \geq N$.

Thus beyond a certain point in the series, any block of consecutive terms, *no matter how long*, must have a very small sum.

In the last section we proved the harmonic series diverges by producing blocks of terms arbitrarily far out in the series whose sum exceeds $\frac{1}{2}$. In other words, we showed that the Cauchy test fails for $\epsilon = \frac{1}{2}$.

Suppose the Cauchy test is satisfied, and take $m = n + 1$. Then the block consists of just one term $a_m$, so $|a_m| < \epsilon$ when $m \geq N$. In other words, $a_m \longrightarrow 0$.

**Necessary Condition for Convergence** If the series $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

WARNING: This condition is not sufficient for convergence. The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges even though $1/n \longrightarrow 0$.

### *Positive Terms*

Suppose an infinite series has only non-negative terms. Then its partial sums form an increasing sequence, $s_1 \leq s_2 \leq s_3 \leq s_4 \leq \cdots$. Recall that an increasing sequence must be one of two types: Either (a) the sequence is bounded above, in which case it converges; or (b) it is not bounded above, and it marches off the map to $+\infty$.

We deduce corresponding statements about series:

A series $a_1 + a_2 + a_3 + \cdots$ with $a_n \geq 0$ converges if and only if there exists a positive number $M$ such that

$$a_1 + a_2 + \cdots + a_n \leq M \qquad \text{for all} \quad n \geq 1.$$

Using this fact, we can often establish the convergence or divergence of a given series by comparing it with a familiar series.

**Comparison Test** Suppose $\sum a_n$ and $\sum b_n$ are series with non-negative terms.

(1) If $\sum a_n$ converges and if $b_n \leq a_n$ for all $n \geq 1$, then $\sum b_n$ also converges.

(2) If $\sum a_n$ diverges and if $b_n \geq a_n$ for all $n \geq 1$, then $\sum b_n$ also diverges.

*Proof:* Let $s_n$ and $t_n$ denote the partial sums of $\sum a_n$ and $\sum b_n$ respectively. Then $\{s_n\}$ and $\{t_n\}$ are increasing sequences.

(1) Since $\sum a_n$ converges, $s_n \leq \sum_1^\infty a_n = M$ for all $n \geq 1$. Since $b_k \leq a_k$ for all $k$, we have $t_n \leq s_n$ for all $n$. Hence $t_n \leq s_n \leq M$ for all $n \geq 1$, so $\sum b_n$ converges.

(2) Since $\sum a_n$ diverges, the sequence $\{s_n\}$ is unbounded. Since $b_k \geq a_k$, we have $t_n \geq s_n$. Hence $\{t_n\}$ is also unbounded, so $\sum b_n$ diverges.

NOTE: It is important to apply the Comparison Test correctly. Roughly speaking, (1) says that "smaller than small is small" and (2) says that "bigger than big is big". However the phrases "smaller than big" and "bigger than small" contain little useful information.