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BANACH ALGEBRAS AND THE GENERAL THEORY OF *-ALGEBRAS

VOLUME I
ALGEBRAS AND BANACH ALGEBRAS

THEODORE W. PALMER

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

*Banach Algebras and The General Theory of *-Algebras*
Volume I: Algebras and Banach Algebras

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This is the first volume of a two volume set that provides a modern account of basic Banach algebra theory including all known results on general Banach *-algebras. This account emphasizes the role of *-algebra structure and explores the algebraic results which underlie the theory of Banach algebras and *-algebras. Both volumes contain previously unpublished results.

This first volume is an independent, self-contained reference on Banach algebra theory. Each topic is treated in the maximum interesting generality within the framework of some class of complex algebras rather than topological algebras.

In both volumes proofs are presented in complete detail at a level accessible to graduate students. In addition, the books contain a wealth of historical comments, background material, examples, particularly in noncommutative harmonic analysis, and an extensive bibliography. Together these books will become the standard reference for the general theory of *-algebras.

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PREFACE

This volume provides a gentle introduction to most of the main areas of research on general Banach algebras. It also serves the more specific purpose of providing the background for Volume II which will deal more intensively with *-algebras (*i.e.* algebras with fixed involutions, normally denoted by *). The focus is on the algebraic, and sometimes the geometric, underpinnings of the analytic theory. The subject is rich with aesthetic appeal, and many topics are pursued just as far as I found them attractive. References are given to more thorough expositions when they are available or to original sources. I have tried to make the book readable for beginning graduate students. Towards this end, I sometimes include a bit of undergraduate level material when it may not have been absorbed by such readers. There are also generous comments and historical remarks. They are all intended to serve a pedagogic purpose. I have tried to document the original source of most ideas, but sometimes I have failed. I apologize to those thus slighted. The knowledgeable reader will also find numerous previously unpublished results and technical improvements.

Readers should note the Symbol Index at the end of the volume. I have chosen notation carefully and used it consistently throughout the work. For instance, $\mathcal{A}$ always represents an algebra and $\mathcal{A}$ with a subscript always represents a subset of that algebra. Each entry in the bibliography displays the numbers of the sections in this volume to which it is related. A few of these entries, primarily those recording recent papers, are not actually referred to in the text but have been included to record the names of current research workers. When it is convenient to state several parallel cases in a single definition or result, the various options are enclosed in angle brackets $\langle\ \rangle$ and separated by an ordinary slash / . Internal references are given by the familiar device of tripartite numbers separated by periods where the $\langle$ first / second / third $\rangle$ number refers to the $\langle$ chapter / section / subsection or statement $\rangle$. Additional notation and conventions are introduced at the beginning of Chapter 1.

In 1970 I began writing a book on *-algebras. Repeatedly I discovered that existing references did not cover the background material in sufficient detail or from the viewpoint I needed. Thus this first volume began life as a series of appendices. Chapter 2 on the spectrum and spectral algebras is a direct descendent of the first of these appendices which attained the status of a complete independent exposition. When the appendices became as long as the main text, I realized that they needed to come first and could easily be expanded to provide a relatively complete introduction to Banach algebra theory. I learned this theory from the ground-breaking book of Charles E. Rickart [1960] and those familiar with his book will see the strong influence his organization of the subject still has on my own.

By 1978 I had written a relatively complete manuscript. Unfortunately it was never quite finished for publication, and I devoted the decade of the 1980's mainly to administrative work. During that whole period, I tried to keep current with work on general Banach algebras, and I continually revised sections and incorporated really striking new results which obviously belonged. However, there was no time to complete the manuscript for publication. A daunting pile of typed pages was the result in those pre-computer days. Finally it became nearly impossible to trace down and change all the cross references when new material was added.

In December 1987 Robert S. Doran (Texas Christian University) asked me how my book was progressing and whether I had a publisher. Wholly involved with dean's work at the time, I replied that I did not see how I would ever finish it without a coauthor. Within a few days he expressed willingness to revise the book as a coauthor and I immediately accepted. Bob quickly arranged a contract with Cambridge University Press, and I began to withdraw from further administrative commitments. In December 1988 Bob sent me a preprint of Thomas J. Ransford's beautiful proof of Barry Johnson's uniqueness of norm theorem. Within a few hours I had used Ransford's method to give a new proof of the fundamental theorem of spectral semi-norms (Theorem 2.3.6). Since I had known for years that an easy early proof of this result was a key to a smooth exposition of many of my ideas, I decided then and there to leave the dean's office and work to complete this book. With Bob's help it seemed a task of two or three years at most. Unfortunately, during the 21 months it took me to free myself of all administrative commitments, Bob was drawn more and more into administration himself. In March 1991 he had to drop his role as coauthor. This book would have greatly benefitted had he been able to continue. Besides securing a contract for publication, Bob worked with his colleague David Addis to develop TeX macros for the book and arranged to have his wife Shirley Doran prepare a preliminary TeX version of my old manuscript. Without all their contributions, I could not have TeXed the whole book. Bob also made numerous suggestions on style. Many readers will thank him for convincing me to give up the use of the Fraktur alphabet. (In my heart, I still believe that $\mathfrak{A}$ is a typical Banach algebra.)

Many other people have helped with completion of this book. All the words and mistakes are my own, but most of the commas were contributed by my wife Laramie and by Kenneth A. Ross, both of whom have proofread essentially the whole work. Richard M. Koch has repeatedly come to my rescue when some computer or TeXnicality defeated me. Laramie has helped with the book from the beginning; Ken and Dick were also essential. John Duncan (University of Arkansas), Robert B. Burckel (Kansas State University) and Michael J. Meyer (Georgia State University) read most of the manuscript, sometimes in earlier versions, and made valuable suggestions. Barry E. Johnson (University of Newcastle) and H. Garth Dales (Leeds University) helped on more limited portions. Numerous colleagues have provided preprints or valuable information. Beginning in 1970 several generations of graduate students at the University of Oregon have seen preliminary versions of the book. They have pointed out obscurities or even errors and in the later stages have contributed to the proofreading efforts. I cannot list them all but here are a few: Abdullah H. Al Moajil, Robert Bekes, Michael Boardman, Sean Bradley, Jon M. Clauss, David Collett, Dan Hendrick, Thomas W. Judson, Michael Leen, Chung Lin, Jorge M. López, Michael Ottinger, William L. Paschke, Paul L. Patterson III, John Phillips, James Rowell, Richard C. Vrem and Sheng L. Wu. To all those mentioned above by name or category, I extend heart-felt appreciation.

With deep filial respect, this volume is dedicated to:

Ernest Jesse Palmer

(April 8, 1875 to February 25, 1962)

and

Elizabeth McDougall Palmer

(March 14, 1902 to April 25, 1972).

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Tentative Contents of Volume II

1

Introduction to Normed Algebras; Examples

Introduction

This chapter begins by stating some basic conventions, definitions and notation that will be used throughout the work. Additional standard notations will be introduced from time to time, as needed. The reader should consult the index of notation for reference. Many of the ideas presented in the first section will be familiar to some readers. They are mentioned for the sake of review and to fix our notation. Also, of course, some standard concepts are defined in slightly different ways by different authors, and we wish to make clear our own conventions. The chapter concludes with a number of examples discussed in some depth. We urge readers to acquaint themselves with these since an abstract theory, such as that presented in this work, lacks substance without knowledge of examples.

The first section deals primarily with basic elementary results on normed, semi-normed, or topological linear spaces and algebras. Such topics as ideals, homomorphisms, quotient norms, etc. are discussed, and the role of semi-norms in locally convex topological linear spaces is quickly surveyed. The unitization of an algebra and an important convention about it are also introduced.

In order to enliven the section, we have introduced several interesting or slightly unusual topics which need relatively few prerequisites. Some basic facts about commutant subsets of algebras and maximal commutative subalgebras are presented. For any submultiplicative semi-norm σ on an algebra $\mathcal{A}$, we present the important properties of the non-negative real-valued function on $\mathcal{A}$ defined by

$$\lim_{n\to\infty} \sigma(a^n)^{1/n} \qquad \forall\, a \in \mathcal{A}.$$

Section 1.2 deals with the double centralizer algebra $\mathcal{D}(\mathcal{A})$ of an algebra $\mathcal{A}$. We regard $\mathcal{D}(\mathcal{A})$ as a more natural unitization of a non-unital algebra $\mathcal{A}$. It also allows the classification of extensions of well-behaved Banach algebras under mild hypotheses. This interesting theory is also presented.

Section 1.3 discusses a number of ways in which algebras and Banach algebras can be combined to make new algebras. It introduces direct sums, direct products, subdirect products, both projective and injective limits, ultraproducts and ultrapowers.

Section 1.4 is devoted to the Arens product. This interesting, fundamental and elementary construction, which provides a product on the double dual Banach space of any Banach algebra, is surveyed in some detail and explored more thoroughly in some of the examples which follow.

The remaining six sections present examples. Some of these will probably be familiar but most are discussed in an elementary fashion to show how the ideas arise naturally and to make them accessible even to the beginner. In many years of teaching this material, we have often noted students who become facile with the theory without knowing a reasonable stock of examples. Thus we urge that these sections be read in detail. Algebras of functions are dealt with very briefly since they belong more to the subject matter of Chapter 3. Matrix algebras, operator algebras and group algebras are presented in more detail.

A number of simple examples, which we might have included here are algebras with involutions. Since the second volume of this book will be devoted to a much deeper study of involutions, we omit most of these examples here. For this reason, most algebras of operators on Hilbert space are omitted or slighted even though they rank among the simplest examples, as will be seen when they are presented in Volume II.

1.1 Norms and Semi-norms on Algebras

Sets, Functions and Notation

If $\mathcal{D}$ and $\mathcal{S}$ are sets, we write $\mathcal{D}\backslash\mathcal{S}$ for the *difference set* $\{a \in \mathcal{D} : a \notin S\}$. If f is any function with domain $\mathcal{D}$, and $\mathcal{S}$ is any set, $f|_S$ denotes the *restriction of* f to the domain $\mathcal{D}\cap\mathcal{S}$. In neither case do we insist that S be a subset of $\mathcal{D}$. We sometimes use the notation $f^{\leftarrow}$ for the relation which is the *inverse* of a function f, particularly when $f^{\leftarrow}$ is not a function. (However if f is a function in an algebra $\mathcal{A}$ of linear functions under composition, we always use f^{-1} for the inverse when it is a function in $\mathcal{A}$. Conversely, if f is a function with values in a group in which multiplicative notation is used or in the invertible elements in some ring, and if f belongs to a group or ring of functions in which multiplication is defined pointwise, then f^{-1} will always represent the function defined by $f^{-1}(x) = f(x)^{-1}$ for each x in the domain of f.) If $\mathcal{F}$ is a set of functions, each with domain including $\mathcal{X}$, we write $\mathcal{F}(\mathcal{X})$ for the set $\{f(x) : f \in \mathcal{F},\, x \in \mathcal{X}\}$.

In any topological space we denote the *boundary, closure* and *interior* of a subset $\mathcal{S}$ by $\partial\mathcal{S}$, $\mathcal{S}^-$, and $\mathcal{S}^\circ$, respectively. The *support* supp$f$ of a real- or complex-valued function f defined on a topological space Ω is the closure of the set where f is non-zero.

We will use $\mathbb{C}$, $\mathbb{D}$, $\mathbb{T}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_+^\bullet$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}^0$ and $\emptyset$ to denote, respectively, the set of *complex numbers,* the *unit disc* $\{\lambda \in \mathbb{C} : |\lambda| \le 1\}$, the 1-*torus*

$\{\lambda \in \mathbb{C} : |\lambda| = 1\}$, the set of *real numbers*, the set of *non-negative real numbers*, the set of *positive real numbers*, the set of all *integers*, the set $\{1, 2, 3, \ldots\}$ of *natural numbers* or *positive integers*, the set $\{0, 1, 2, \ldots\}$ of *non-negative integers* and the *empty set*. We endow each of these sets with all its usual structure so that, for instance, $\mathbb{R}$ is an ordered normed field. *Open* and *closed intervals* are denoted by $]\cdot,\cdot[$ and $[\cdot,\cdot]$, respectively. The *supremum* of a set in $\mathbb{R}$ is ∞ if the set is unbounded and $-\infty$ if the set is empty. Similar conventions hold for the *infimum*. The *complex conjugate* of a complex number λ will be denoted by λ^*. We frequently use the *Kronecker delta*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The indices i and j may be any type of mathematical object.

It is often convenient to state several definitions or results in a parallel fashion. We do so by listing the various choices involved in order, enclosed in angular brackets and separated by slashes: $\langle \ldots / \ldots / \ldots \rangle$. References within this work are given by a three part number made up of the chapter number, section number and result or subsection number. References to other books and articles are given by mentioning the author's name in the text and then giving the year of publication enclosed in square brackets. Full details are located in the bibliography.

Algebras and Subalgebras

1.1.1 Definition An *algebra* $\mathcal{A}$ over a field $\mathbb{F}$ is a ring which is also a linear space over $\mathbb{F}$ under the same addition and satisfies

$$(\lambda a)b = \lambda(ab) = a(\lambda b) \qquad \forall\, \lambda \in \mathbb{F};\ a,\, b \in \mathcal{A}.$$

N. B. Throughout this work all linear spaces or algebras will have the **complex** field as their field of scalars unless the contrary is explicitly stated. Occasionally $\mathbb{F}$ will denote either the real or the complex field: $\mathbb{R}$ or $\mathbb{C}$.

An algebra is said to be *unital* if it has a multiplicative identity (*i.e.*, an element 1 satisfying $1a = a1 = a$ for all $a \in \mathcal{A}$).

A *subalgebra* is a subring which is also a linear subspace (*i.e.* it is a linear subspace which contains the product of any of its elements). A subalgebra of a unital algebra which contains the multiplicative identity element of the larger algebra is called a *unital subalgebra.*

We will always state definitions and results for algebras even when the theory for arbitrary rings is no more complicated. When feasible, we will try to state definitions, propositions and arguments so that they also apply to algebras with the real numbers as scalar field but this is not always possible. Note that a subalgebra may be a unital algebra (because it contains its own identity element) without being a unital subalgebra (because the larger algebra is either nonunital or has a different identity element).

An element a in an algebra $\mathcal{A}$ is called an $\langle$ *idempotent* / *nilpotent* $\rangle$ if it satisfies $\langle\ a^2 = a\ /\ a^n = 0$ for some $n \in \mathbb{N}\ \rangle$. An idempotent is said to be *proper* if it is not a multiplicative identity element for the algebra to which it belongs. A non-zero proper idempotent is said to be *nontrivial.* Finally, two idempotents e and f are *orthogonal* if they satisfy $ef = fe = 0$.

An element a in an algebra $\mathcal{A}$ is called a $\langle$ *left* / *right* $\rangle$ *divisor of zero* if there is some non-zero $b \in \mathcal{A}$ satisfying $\langle\ ab = 0\ /\ ba = 0\ \rangle$. An element which is either a left or right divisor of zero is called a *divisor of zero* and an element which is both is called a *two-sided divisor of zero* and a *joint divisor of zero* if the same element b can be used on both sides.

Historical Remarks on Algebras

The term "algebra", was first applied, in the sense used in this work, by Benjamin Peirce [1870]. His interesting paper, which was published posthumously in 1881, was in part a philosophical discussion intended to establish the modern 20th century view that "Mathematics is the science which draws necessary conclusions." As an example, he defined "linear associative algebras", by axioms and derived a number of consequences. For another third of a century most mathematicians studied algebras as "systems of hypercomplex numbers". This term denoted a linear space with a given distinguished basis and a multiplication table for the basis elements giving each product as a specified linear combination of basis elements. Shortly after the turn of the century, under the influence of Leonard Eugene Dickson, the term "algebra", and the axiomatic definition and viewpoint were generally adopted by mathematicians. For instance, at Dickson's suggestion the term "algebra", (but not the axiomatic definition) is used in the text of Joseph H. Maclagan Wedderburn's classic paper [1907] establishing his decomposition theorems. Dickson used both the term "algebra", and the axiomatic definition in his influential book [1927]. See also Karen Hunger Parshall [1985].

Linear Spans, Convex Hulls, Products and Sums of Subsets

When $\mathcal{S}$ is a subset of a linear space $\mathcal{X}$ we write span($\mathcal{S}$) for the linear span of $\mathcal{S}$ and co($\mathcal{S}$) for the convex hull of $\mathcal{S}$. If $\mathcal{X}$ is the linear span of a linearly independent set, we call the set a *Hamel basis* for $\mathcal{X}$. A subset $\mathcal{S}$ of a linear space is said to be *balanced* if λx belongs to $\mathcal{S}$ for all complex λ satisfying $|\lambda| \le 1$ and all $x \in \mathcal{S}$. The *balanced convex hull* of a subset $\mathcal{S}$ of $\mathcal{X}$, denoted by ba($\mathcal{S}$), is the set

$$\left\{\sum_{j=1}^{n} \lambda_j x_j : n \in \mathbb{N};\ \lambda_j \in \mathbb{C},\ x_j \in \mathcal{S} \text{ for } j = 1, 2, \ldots, n;\ \sum_{j=1}^{n} |\lambda_j| \le 1\right\}.$$

Let $\mathcal{A}$ be an algebra. If $\mathcal{S}$ and $\mathcal{T}$ are linear subspaces of $\mathcal{A}$, we will denote the set span$\{ab : a \in \mathcal{S},\ b \in \mathcal{T}\}$ by $\mathcal{ST}$. Also if a is an element of $\mathcal{A}$ and $\mathcal{S}$ and