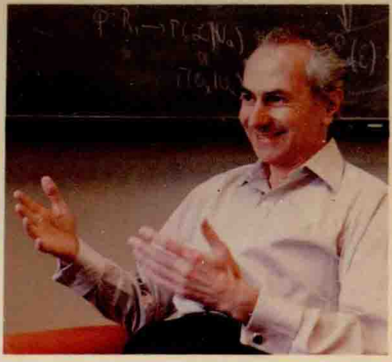
Serge Lang



MATH!

Encounters with High School Students

Springer-Verlag New York Berlin Heidelberg Tokyo

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With 103 Illustrations

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MATH!

Acknowledgement

I want to thank the teachers who made it possible for me to meet their classes: William Bisset, Patricia Chwat, Peter Edwards, Marie-Therese Giacomo, Michel Ricart. I know that there are some bad teachers, but there are also some good ones, and the students with whom I did mathematics would certainly not have reacted as they did if their teachers had been bad. I thank Abe Shenitzer, who induced me to give the talks in Canada, and organized them. I also thank Jean Brette, who directs the Mathematics Section of the Palais de la Découverte in Paris, for the contacts which arose through him with high school teachers in Paris, and also for his interest in the whole pedagogical enterprise. I also appreciated Stephane Brette's interest, and his willingness to take part in a mathematical dialogue with me, after one of the talks. I thank Patrick Huet who video-taped for the Paris IREM two of the talks, and the discussion reproduced at the end of this book. Finally I thank Carol MacPherson for the photograph on the cover.

SERGE LANG

Who is Serge Lang?

Serge Lang was born in Paris in 1927. He went to school until the 10th grade in the suburbs of Paris, where he lived. Then he moved to the United States. He did two years of high school in California, then entered the California Institute of Technology (Caltech), from which he graduated in 1946. After a year and a half in the American army, he went to Princeton in the Philosophy Department where he spent a year. He then switched to mathematics, also at Princeton, and received his PhD in 1951. He taught at the university and spent a year at the Institute for Advanced Study, which is also in Princeton.

Then he got into more regular positions: Instructor at the University of Chicago, 1953–1955; Professor at Columbia University, 1955–1970. In between, he spent a year as a Fulbright scholar in Paris in 1958.

He left Columbia in 1970. He was Visiting Professor at Princeton in 1970–1971, and Harvard in 1971–1972. Since 1972 he has been a professor at Yale.

Besides math, he mostly likes music. During different periods of his life, he played the piano and the lute.

From 1966 to 1969, Serge Lang was politically and socially active, during a period when the United States faced numerous problems which affected the universities very deeply.

He has also been concerned with the problems of financing the universities, and of their intellectual freedom, threatened by political and bureaucratic interference. As he says, such problems are invariant under ism transformations: socialism, communism, capitalism, or any other ism in the ology.

However, his principal interest has always been for mathematics. He has published 28 books and more than 60 research articles. He received the Cole Prize in the U.S. and Prix Carriere in France.

Dear Christopher, Rachel, Sylvain, Yaelle, and all the others

I am writing this to you because I came to schools like yours, in France as well as in Canada, to talk mathematics with students who could have been your own friends, and who thus contributed to a joint enterprise.

I wanted to show them beautiful mathematics, at the level of your class, but conceived the way a mathematician does it. In most school books, the topics are usually treated in a way which I find incoherent. They pile up one little thing on another, without rhyme or reason. They accumulate technical details endlessly, without showing the great lines of thought in which technique can be inserted, so that it becomes both appealing and meaningful. They don't show the great mathematical lines, similar to musical lines in a great piece of music. And it's a great pity, because to do mathematics is a lively and beautiful activity.

This book is made up of several lectures, or rather dialogues, which have been transcribed as faithfully as possible from the tapes, to preserve their lively style. It gave me great pleasure to have this kind of exchange with all the students, in different classes. The subjects concern geometric and algebraic topics, understandable at the ninth and tenth grade level. I even gave one of the talks to an eighth grade class! If students at those levels could understand and enjoy the mathematics involved, so can you.

Each dialogue is self-contained, so you don't have to read this book continuously from beginning to end. Each topic forms a single unit, which you can enjoy independently of the others. If, while reading any one of them, you find the going too rough, don't let that put you off. Keep on reading, skip those passages that don't sink in right away, and you will probably find something later in the lecture which is easier and more accessible to you. If you are still interested, come back to those passages which gave you trouble. You will be surprised how often, after sleeping on it, something which appeared hard suddenly becomes easy. Just browse through the book, pick and choose, and mostly get your mind to function.

A lot of the curriculum of elementary and high schools is very dry. You may never have had the chance to see what beautiful mathematics is like. I hope that if you are a high school student, you will be able to complement whatever math course you are taking by reading through this book.* I have many objections to the high school curriculum. Perhaps the main one is the incoherence of what is done there, the lack of sweep, the little exercises that don't mean anything. You will find something quite different here, which I hope will inspire you. And by doing mathematics, you might end up by liking math as you like music, or as I like it.

SERGE LANG

* The first five talks would fit well in a geometry course, and the last two in an algebra course. See also the book which I wrote in collaboration with Gene Murrow: *Geometry*, published by Springer-Verlag.

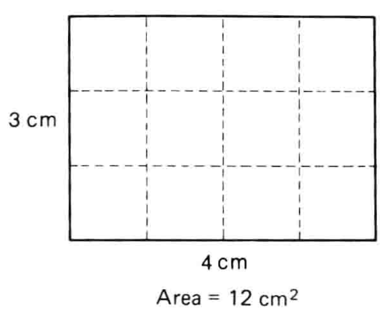
Contents

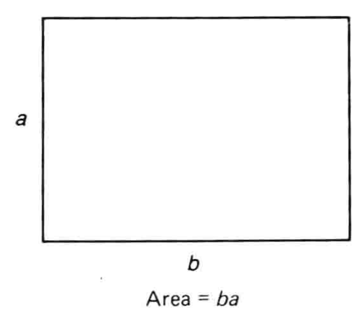
What is pi?

The following talk was given at a high school in the suburbs of Toronto, April 1982, to a class of students about 15 years old. The talk lasted about 1 hour and 15 minutes.

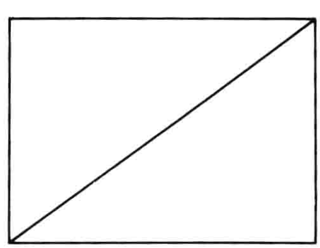
SERGE LANG. My name is Serge Lang, I usually teach at Yale, but today I came here to do mathematics with you.

We are going to study the area of some simple geometric objects, like rectangles, triangles, and circles which you must have heard about in this course. Let's start with rectangles. We assume that its area is the product of the base times the height, so if the sides have lengths a and b, then the area of the rectangle is ab. For instance, if a rectangle has sides of lengths 3 cm and 4 cm, then its area is 12 cm^2. You can check this on the figure.





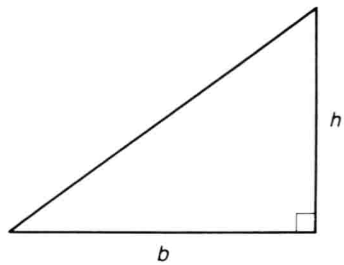
If you cut a rectangle in half, like this:



then you get a right triangle, so the area of the right triangle is one half the product of the base times the height. We can write

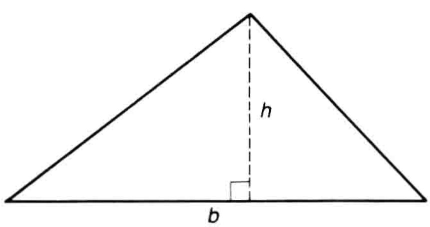
$$\text{area of right triangle} = \frac{1}{2}bh$$

where b is the base and h is the height.

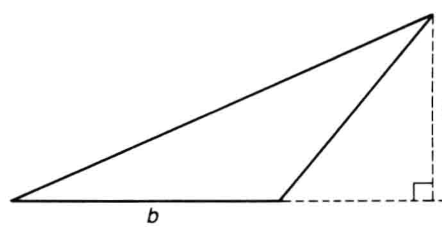


Area $= \frac{1}{2}bh$

You should also know that this formula is true for any triangle, if h is the perpendicular height. I can show you this on two possible figures:



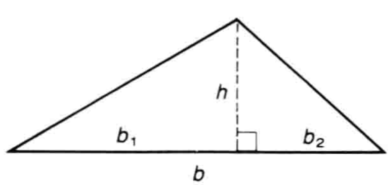
Area $= \frac{1}{2}bh$

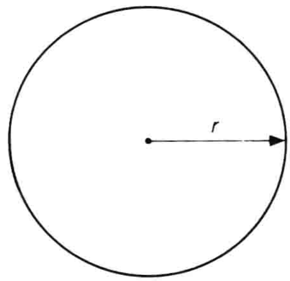


Area $= \frac{1}{2}bh$

Try to prove the formula yourself, because I want to have time to discuss something more interesting, the circle.[1] So we assume you know about the area of a triangle. Now you have the circle radius r.

[1] I give the proof for the first figure. Drop the perpendicular height from one vertex to the opposite side as shown on the next figure.





Do you know what is the formula for its area?

A STUDENT. It's pi r squared.

SERGE LANG. That's right, it's πr^2. Well, what is pi?

A STUDENT. What is pi?

SERGE LANG. Yes.

THE STUDENT. 3.14

SERGE LANG. You claim that π is 3.14. Is that an exact expression?

THE STUDENT. No, I don't think so.

SERGE LANG. Then why did you say 3.14?

STUDENT. Well, it goes on and on.

SERGE LANG. OK. So we put dot dot dot like this, 3.14... to signify that it goes on and on and on. How do you know how it goes on?

[*The students react variously.*]

SERGE LANG. It's not so clear! So it's a problem how it goes on. I mean, how are you going to compute it?

A STUDENT. You measure the perimeter of the circle and divide by twice the radius.

SERGE LANG. Ah, now you're telling me something else. Instead of the perimeter, let me call it the circumference. Do you mind if I call it the circumference?

STUDENT. No.

Then the triangle is decomposed into two right triangles, whose bases are b_1 and b_2 such that $b_1 + b_2 = b$. The two right triangles have the same height h. Then using the formula for the area of a right triangle, we now get:

$$\text{area of the triangle} = \frac{1}{2}b_1 h + \frac{1}{2}b_2 h = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}bh.$$

This proves the formula for the first figure. Treat the second figure similarly. You will need a subtraction instead of an addition.

SERGE LANG. OK. So first you told me the area is πr^2, and now you mention the circumference. We call the circumference c. And what did you just tell me? You made the assertion that

$$c = 2\pi r.$$

That's what you said: $2r$ is the diameter, so the circumference is pi times the diameter, so we can also write

$$c = \pi d,$$

where d is the diameter, $d = 2r$. But now look. You have two formulas, for the area and for the circumference:

$$\pi r^2 \quad \text{and} \quad 2\pi r.$$

By the way, what's your name?

STUDENT. Serge.

SERGE LANG. Oh Serge, just like me! [*Laughter.*] Serge said, to compute π, you look at the circumference and divide by the diameter. The circumference is something you can measure. You can get a soft tape at home, you put it around a frying pan, and you measure the circumference. Then you measure the diameter with a ruler, and divide. Actually you can get one or two decimals accuracy out of that, probably you can get two decimals if you are careful. You get some sort of value, which is an approximation for π.

You would have a much harder time trying to measure the area to get an approximate value for π.

Now the question is: you've got these two formulas, one for the area, one for the circumference. How do you know these formulas are true?

STUDENTS. [*Silence, questioning looks.*]

SERGE LANG. How do you prove them? Has anybody ever broached the problem of proving these formulas? At any time? You were just given the formulas.

STUDENTS. [*Negative looks on most faces, one or two raise their hands.*]

SERGE. You can just say that π equals the circumference divided by the diameter, and work it out.

SERGE LANG. Work what out? You just repeated one of the two formulas. You have two formulas. Suppose I want to prove them. To prove them I have to start from something and then I have to get to the formulas by logic. So I start from what?

SERGE. You start from where you divide the circumference by the diameter, that equals π.

SERGE LANG. And then what? Now you have to reach the area. What is the definition of π? Before you can prove something, you must have a definition.

SERGE. It's what I said, the circumference divided by the diameter.

SERGE LANG. But then, you have to show that it's the same π in the formula for the area. If you tell me that π is the circumference divided by the diameter, which is twice the radius, you can start with that as a definition, but then you have to prove something, which is the other formula.[2]

So we have to start from something, with a definition, otherwise I can't prove anything. And then logically, derive the formulas. So the question is, where do we start from? That's what I am after. I want to start from somewhere, and get to these two formulas.

I will have to explain two things about these formulas. One is where the r^2 comes from; and second, where the π comes from. They come from two different aspects of the problem. One of the aspects has to do with the r^2. Why is there an r^2 in the formulas for the area? And why is there an r (but not r^2) for the circumference? The presence of the r and r^2 has to be discussed. And the other thing I have to dicuss is the π.

So we start all over. I'll first explain the r^2, and after that I'll explain the π.

Let's go back to the even simpler case of the rectangle. Suppose I have a rectangle of sides a, b. Then the area of the rectangle is just the product, ab. Now suppose I take a rectangle whose sides are twice a and twice b, so I blow up the rectangle by a factor of 2. How does the area change?

A STUDENT. It doubles.

SERGE LANG. What's your name?

THE STUDENT. Adolph.

SERGE LANG. The area doubles? What is the area of the new rectangle?

[*Another student starts talking.*]

SERGE LANG. No. Adolph, I am asking Adolph. The area of a rectangle is the product of the sides. Right?

ADOLPH. Yes.

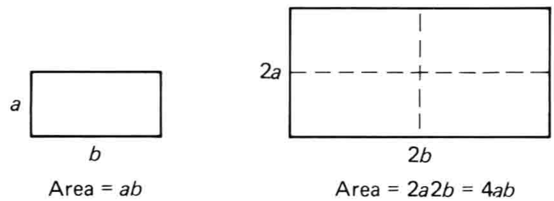
SERGE LANG. So one side of the new rectangle is $2a$, and what is the other side?

ADOLPH. It's $2b$.

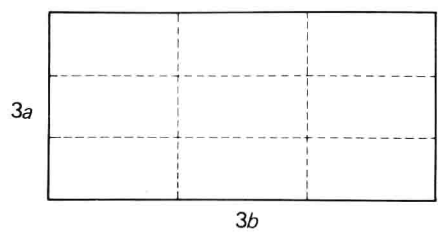
---

[2] You also have to prove that no matter what circle you take, the ratio of the circumference by the diameter gives the same number. This is precisely one of the things we are trying to prove. I was not on the ball when I did not raise this objection explicitly that way.

SERGE LANG. That's right, so the total area is $2a$ times $2b$, which is $4ab$.



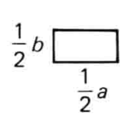
Now suppose I take a rectangle with three times the sides. So here I have sides of lengths $3a$ and $3b$.



What is the area of the rectangle with three times the sides? Adolph.

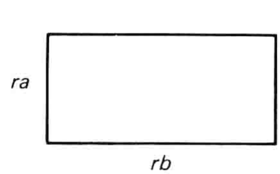
ADOLPH. $9ab$.

SERGE LANG. That's right, $9ab$, it's $3a$ times $3b$, which is $9ab$. Suppose I now take a rectangle with one half the sides, so I have here $\frac{1}{2}a$, and $\frac{1}{2}b$. What is the area of this rectangle, with one half the sides?



ADOLPH. It's *ab* over 4.

SERGE LANG. That's right. One fourth *ab*. Now suppose in general I take a rectangle with the sides *ra* and *rb*. Like this.



Adolph, what's the area of this rectangle?

ADOLPH. *ra* times *rb*.

SERGE LANG. That's right, *ra* times *rb*, which is what?

ADOLPH. r squared times *ab*.

SERGE LANG. Yes, r^2ab. So if I change the rectangle by a blow up by a factor of r, how does the area change? Adolph.

ADOLPH. Can you repeat that, please?

SERGE LANG. Yes. I have my old rectangle. I blow it up by a factor of r. In both directions. You see, the new sides are *ra*, *rb*? How does the area change? The old area was *ab*. What is the new area?

ADOLPH. r squared *ab*.

SERGE LANG. Yes, r^2ab, not *rab*. The area changes, by what factor?

ADOLPH. By r^2.

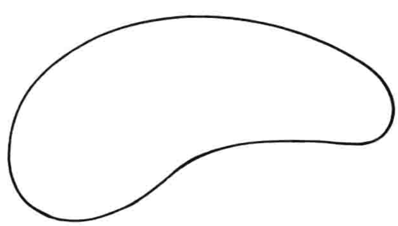
SERGE LANG. You said r a minute ago. Well it's not r. It's r^2. You see it? Does everybody see it?

[*Students agree that they see it.*]

SERGE LANG. So if I make a blow up by a factor of r, the area of a rectangle changes by a factor of r^2. And of course, r can be bigger than 1, or r can be smaller than 1, like $r = 1/2$ or $r = 1/3$. So now you see how area changes for rectangles. Any questions? Everybody got that?

[*No questions.*]

All right, now instead of a rectangle, suppose I have another figure. Suppose I have a curved figure, like this.

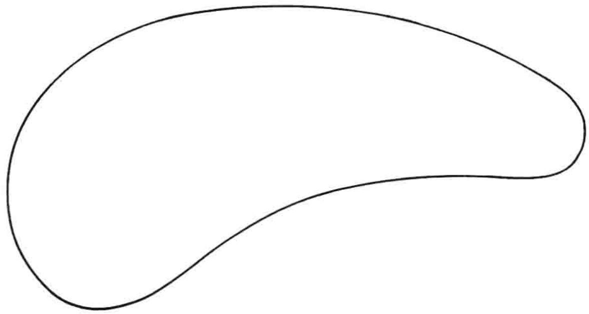


Just like a kidney. Suppose I have a kidney, and it has a certain area A. Now I blow up the kidney by a factor of 2, for example. What will be the area of the blown up kidney?

A STUDENT. A^2?

SERGE LANG. No, let's go back. I have a rectangle with area A. I blow this rectangle up by a factor of 2. What is the new area?

A STUDENT. It's $4A$.



SERGE LANG. Yes, the area changes by 2 squared. If I blow up the rectangle by a factor of r, the area changes by a factor of r^2. Suppose I blow up a curved figure. I blow up by a factor of 2. The area will change by what factor?

A STUDENT. 4.

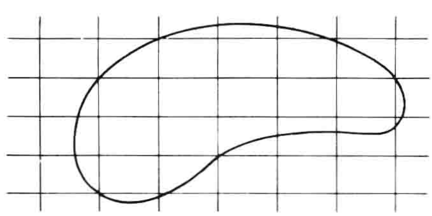
SERGE LANG. OK, the area changes by this same factor of 4. Why? What is the proof? I have a curved figure, a kidney, not a rectangle. How do I prove it? Anybody have any ideas? All right, Adolph.

ADOLPH. You measure around.

SERGE LANG. No, you don't determine area by measuring around. If I measure around, I get the perimeter, the circumference. I am now dealing with the area. The whole thing inside.

[*A student raises her hand. There is a certain amount of fumbling around by several students. After a while, Serge Lang picks up again.*]

SERGE LANG. I try to reduce the question to rectangles. I make a grid like this.



You see the grid? Then the area of the kidney is approximated by the area of the rectangles which are inside the kidney. I look at all the rectangles here which lie completely inside the kidney. [*Serge Lang draws the thick line in the next figure.*] They go like that, all the way down there, up here, there, and here.