Analytic Perturbation Theory and Its Applications

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siam.
Society for Industrial and Applied Mathematics
Philadelphia

10 9 8 7 6 5 4 3 2 1

Library of Congress Cataloging-in-Publication Data

Avrachenkov, Konstantin, author.
Analytic perturbation theory and its applications / Konstantin E. Avrachenkov, Inria Sophia Antipolis, Sophia Antipolis, France, Jerzy A. Filar, Flinders University, Adelaide, Australia, Phil G. Howlett, University of South Australia, Adelaide, Australia.
pages cm
Includes bibliographical references and index.
ISBN 978-1-611973-13-6
1. Perturbation (Mathematics) I. Filar, Jerzy A., 1949- author. II. Howlett, P. G. (Philip G.), 1944- author. III. Title.
QA871.A97 2013
515′.392--dc23
2013033335

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To our students, who, we believe,
will advance this topic far beyond
what is reported here.
Though they may not realize it,
we learned from them at least as much
as they learned from us.

Preface

We live in an era in which ever more complex phenomena (e.g., climate change dynamics, stock markets, complex logistics, and the Internet) are being described with the help of mathematical models, frequently referred to as systems. These systems typically depend on one or more parameters that are assigned nominal values based on the current understanding of the phenomena. Since, usually, these nominal values are only estimates, it is important to know how deviations from these values affect the solutions of the system and, in particular, whether for some of these parameters even small deviations from nominal values can have a big impact.

Naturally, it is crucially important to understand the underlying causes and nature of these big impacts and to do so for neighborhoods of multiparameter configurations. Unfortunately, in their most general settings, multiparameter deviations are still too complex to analyze fully, and even single-parameter deviations pose significant technical challenges. Nonetheless, the latter constitute a natural starting point, especially since in recent years much progress has been made in analyzing the asymptotic behavior of these single-parameter deviations in many special settings arising in the sciences, engineering, and economics.

Consequently, in this book we consider systems that can be disturbed, to a varying degree, by changing the value of a single perturbation parameter loosely referred to as the "perturbation." Since in most applications such a perturbation would be small but unknown, a fundamental issue that needs to be understood is the behavior of the solutions as the perturbation tends to zero. This issue is important because for many of the most interesting applications there is, roughly speaking, a discontinuity at the limit, which complicates the analysis. These are the so-called singularly perturbed problems.

Put a little more precisely, the book analyzes—in a unified way—the general linear and nonlinear systems of algebraic equations that depend on a small perturbation parameter. The perturbation is analytic; that is, left-hand sides of the perturbed equations can be expanded as a power series of the perturbation parameter. However, the solutions may have more complicated expansions such as Laurent or even Puiseux series. These series expansions form a basis for the asymptotic analysis (as the perturbation tends to zero). The analysis is then applied to a wide range of problems including Markov processes, constrained optimization, and linear operators on Hilbert and Banach spaces. The recurrent common themes in the analyses presented is the use of fundamental equations, series expansions, and the appropriate partitioning of the domain and range spaces.

We would like to gratefully acknowledge most valuable contributions from many colleagues and students including Amie Albrecht, Eitan Altman, Vladimir Ejov, Vladimir Gaitsgory, Moshe Haviv, Jean-Bernard Lasserre, Nelly Litvak, (the late) Charles Pearce, and Jago Korf. Similarly, the institutions where we have worked during the long period of writing, University of South Australia, Inria, and Flinders University, have also generously supported this effort. Finally, many of the analyses reported here were carried out as parts of Discovery and International Linkage grants from the Australian Research Council.

Konstantin E. Avrachenkov, Jerzy A. Filar, and Phil G. Howlett

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Chapter 1

Introduction and Motivation

1.1 ▪ Background

In a vast majority of applications of mathematics, systems of governing equations include parameters that are assumed to have known values. Of course, in practice, these values may be known only up to a certain level of accuracy. Hence, it is essential to understand how deviations from their nominal values may affect solutions of these governing equations. Naturally, there is a desire to study the effect of all possible deviations. However, in its most general setting, this is a formidable challenge, and hence structural assumptions are usually required if strong, constructive results are to be explicitly derived.

Frequently, parameters of interest will be coefficients of a matrix. Therefore, it is natural to begin investigations by analyzing matrices with perturbed elements. Historically, there was a lot of interest in understanding how such perturbations affect key properties of the matrix. For instance, how will the eigenvalues and eigenvectors of this matrix be affected?

Perhaps the first comprehensive set of answers was supplied in the, now classical, treatise of Kato [99]. Indeed, Kato's treatment was more general and covered the analysis of linear operators as well as matrices. However, Kato [99] and a majority of other researchers have concentrated their effort on the perturbation analysis of the eigenvalue problem.

In this book we shall study a range of problems that is more general than spectral analysis. In particular, we will be interested in the behavior of solutions to perturbed linear and polynomial systems of equations, perturbed mathematical programming problems, perturbed Markov chains and Markov decision processes, and some corresponding extensions to operators in Hilbert and Banach spaces.

In the same spirit as Kato, we focus on the case of analytic perturbations. The latter have the structural form where the perturbed data specifying the problem can be expanded as a power series in terms of first, second, and higher orders of deviations multiplied by corresponding powers of an auxiliary perturbation variable. When that variable tends to zero the perturbation dissipates and the problem reduces to the original, unperturbed, problem. Nonetheless, the same need not be true of the solutions that are of most interest to the researchers studying the system. These can exhibit complex behaviors that involve discontinuities, singularities, and branching.

Indeed, since the 1960s researchers in various disciplines have studied particular manifestations of the complex behavior of solutions to many important problems.

For instance, perturbed mathematical programs were studied by Pervozvanski and Gaitsgori [126], and the study of perturbed Markov chains was, perhaps, formally initiated by Schweitzer [137]. It is this, not uncommon, complexity of the limiting behavior of solutions that stimulated the present book.

1.2 ▪ Raison d'Être and Exclusions

Imagine that the perturbed matrix mentioned in the previous section had the form

$$\tilde{A} = A + D, \tag{1.1}$$

where A is a matrix of nominal coefficient values, $\tilde{A}$ is a matrix of perturbed data, and D is the perturbation itself. There are numerous publications devoted to this subject (see, e.g., the books by Stewart and Sun [147] and Konstantinov et al. [103] and the survey by Higham [80]). However, without any further structural assumptions on D, asymptotic analysis as the norm of D tends to zero is typically only possible when the rank of the perturbed matrix $\tilde{A}$ is the same as the rank of A. Roughly speaking, this corresponds to the case of what we later define to be a *regular perturbation*. Generally, in such a case solutions of the perturbed problem tend to solutions of the original unperturbed problem.

In this book we wish to explain some of the complex asymptotic behavior of solutions such as discontinuity, singularity, and branching. Typically, this arises when the rank of the perturbed matrix $\tilde{A}$ is different from the rank of A. For instance, consider the simple system of linear equations

$$\tilde{A}x = \begin{bmatrix} 1 & 1 \\ 1+\varepsilon & 1+2\varepsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{1.2}$$

Clearly, $\tilde{A}$ is of the form (1.1) since we can write

$$\tilde{A} = A + D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \varepsilon & 2\varepsilon \end{bmatrix}.$$

Now, for any $\varepsilon \neq 0$, the inverse of $\tilde{A}$ exists and can be written as

$$\tilde{A}^{-1} = \frac{1}{\varepsilon}\begin{bmatrix} 1+2\varepsilon & -1 \\ -1-\varepsilon & 1 \end{bmatrix} = \frac{1}{\varepsilon}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}.$$

Hence, the unique solution of (1.2) has the form of *Laurent series*

$$\tilde{x} = \frac{1}{\varepsilon}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Despite the fact that the norm of D tends to 0 as $\varepsilon \to 0$, we see that $\tilde{x}$ diverges. The singular part of the Laurent series indicates the direction along which $\tilde{x}$ diverges as $\varepsilon \to 0$.

The above example indicates that a singularity manifests itself in the series expansion of a solution. This phenomenon is common in a wide range of interesting mathematical and applied problems and lends itself to rigorous analysis if we impose the additional assumption that the perturbed matrix is of the form

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \dots, \tag{1.3}$$

where the above power series is assumed to be convergent in some neighborhood of $\varepsilon = 0$. Hence it is natural to call this particular type of perturbation an *analytic perturbation*.

Consequently, it is also natural to consider a *singular perturbation* to be one where solutions to the perturbed problem are not analytic functions with respect to the perturbation parameter ε.

It will be seen that with the above analytic perturbation assumption, a unified treatment of both the regular and singular perturbations is possible. Indeed, the approach we propose has been inspired by Kato's systematic analysis of the perturbed spectrum problem but applied to a much wider class of problems. Thus, while Kato's motivating problem is captured by the eigenvalue equation

$$A(\varepsilon)x(\varepsilon) = \lambda(\varepsilon)x(\varepsilon), \tag{1.4}$$

our motivating problem is the asymptotic behavior of solutions to the perturbed system of equations

$$f(x,\varepsilon) = 0,$$

where $f(x,\varepsilon)$ can be a system of linear or polynomial equations. In the linear case this reduces to

$$L(\varepsilon)x(\varepsilon) = c(\varepsilon).$$

In particular, if $L(\varepsilon)$ has an inverse for $\varepsilon \neq 0$, and sufficiently small, then we investigate the properties of the perturbed inverse operator $L^{-1}(\varepsilon)$ (or matrix-valued function $A^{-1}(\varepsilon)$ in the finite dimensional case). For example, we rely on the fact that $A^{-1}(\varepsilon)$ can always be expanded as a Laurent series

$$A^{-1}(\varepsilon) = \frac{1}{\varepsilon^s}B_{-s} + \cdots + \frac{1}{\varepsilon}B_{-1} + B_0 + \varepsilon B_1 + \ldots . \tag{1.5}$$

The preceding system equation $f(x,\varepsilon) = 0$ arises as a building block of solutions to many practical problems. In particular, there is an enormous number of problems that are formulated as either linear or nonlinear mathematical programs. Hence a fundamental question that arises concerns the stability (or instability) of a solution when the problem is slightly perturbed.

Perhaps surprisingly, this can be a very difficult question. Even in the simplest case of linear programming, standard Operations Research textbooks discuss only the most straightforward cases and scrupulously avoid the general issue of how to analyze the effect of a perturbation when the whole coefficient matrix is also affected.

The next example (taken from [126]) illustrates that even in the "trivial" case of linear programming the effect of a small perturbation can be "nontrivial." Consider the simple optimization problem in two variables

$$\max_{x_1,x_2} \; x_2$$

$$\begin{array}{lrcrcl} \text{s.t.} & x_1 & + & x_2 & = & 1, \\ & (1+\varepsilon)x_1 & + & (1+2\varepsilon)x_2 & = & 1+\varepsilon, \end{array}$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

It is clear that for any $\varepsilon > 0$ there is a unique (and hence optimal) feasible solution at $x_1^* = 1$, $x_2^* = 0$. However, when $\varepsilon = 0$, the two equality constraints coincide, the set of feasible solutions becomes infinite, and the maximum is attained at $\hat{x}_1 = 0$, $\hat{x}_2 = 1$.

More generally, techniques developed in this book permit us to describe the asymptotic behavior of solutions[1] to a generic, perturbed, mathematical program:

[1]The word *solution* is used in a broad sense at this stage. In some cases the solution will, indeed, be a global optimum, while in other cases it will be only a local optimum or a stationary point.

$$\begin{aligned} &\max f(x,\varepsilon) \\ \text{s.t.} \quad & g_i(x,\varepsilon)=0,\ i=1,\ldots,m, \\ & h_j(x,\varepsilon)\le 0,\ j=1,\ldots,p, \end{aligned} \tag{MP(ε)}$$

where $x \in \mathbb{R}^n, \varepsilon \in [0,\infty)$, and f, g_i's, h_j's are functions on $\mathbb{R}^n \times [0,\infty)$. We will be especially concerned with characterizing solutions, $x^*(\varepsilon)$, of (MP(ε)) as functions of the perturbation parameter, ε. This class of problems is closely related to the well-established topics of sensitivity or postoptimality, or parametric analysis of mathematical programs (see Bonnans and Shapiro [29]). However, our approach covers both the regularly and singularly perturbed problems and thereby resolves instances such as that illustrated in the above simple linear programming example.

Other important applications treated here include perturbed Markov chains and decision processes and their applications to Google PageRank and the Hamiltonian cycle problems.

Let us give an idea of applicability of the perturbation theory to the example of Google PageRank. PageRank is one of the principal criteria according to which Google sorts answers to a user's query. It is a centrality ranking on the directed graph of web pages and hyperlinks. Let A be an adjacency matrix of this graph. Namely, $a_{ij}=1$ if there is a hyperlink from page i to page j, and $a_{ij}=0$ otherwise. Let D be a diagonal matrix whose diagonal elements are equal to the out-degrees of the vertices. The matrix $L=D-A$ is called the graph Laplacian. If a page does not have outgoing hyperlinks, it is assumed that it points to all pages. Also, let v^T be a probability distribution vector which defines preferences of some group of users, and let ε be some regularization parameter. Then, PageRank can be defined by the following equation:

$$\pi = \varepsilon v^T [L+\varepsilon A]^{-1} D.$$

Since the graph Laplacian L has at least one zero eigenvalue, $L+\varepsilon A$ is a singular perturbation of L, and its inverse can be expressed in the form of Laurent series (1.5). This application is studied in detail in Chapter 6.

Consequently, the book is intended to bridge at least some of the gap between the theoretical perturbation analysis and areas of applications where perturbations arise naturally and cause difficulties in the interpretation of "solutions" which require rigorous and yet pragmatic resolution. To achieve this goal, the book is organized as an advanced textbook rather than a research monograph. In particular, a lot of expository material has been included to make the book as self-contained as practicable. In the next section, we outline a number of possible courses that can be taught on the basis of the material covered. Nonetheless, the book also contains sufficiently many new, or very recent, results to be of interest to researchers involved in the study of perturbed systems.

Finally, it must be acknowledged that a number of, clearly relevant, topics have been excluded so as to limit the scope of this text. These include the theories of perturbed ordinary and partial differential equations, stochastic diffusions, and perturbations of the spectrum. Most of these are well covered by several existing books such as Kato [99], Baumgärtel [22], O'Malley [125], Vasileva et al. [153], Kevorkian and Cole [102], and Verhulst [156]. Singular perturbations of Markov processes in continuous time are well covered in the book of Yin and Zhang [162]. Elementwise regular perturbations of matrices are extensively treated in the books of Stewart and Sun [147] and Konstantinov et al. [103].

Although the question of numerical computation is an extremely important aspect of perturbation analysis, we shall not undertake per se a systematic study of this topic.