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Demetrios Christodoulou

The Formation of Shocks in 3-Dimensional Fluids

European Mathematical Society

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2000 Mathematical Subject Classification (primary; secondary):
35L67; 35L65, 35L70, 58J45, 76L05, 76N15, 76Y05

ISBN 978-3-03719-031-9

Bibliographic information published by Die Deutsche Bibliothek

Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliographie; detailed bibliographic data are available in the Internet at http://dnb.ddb.de.

Contact address:

European Mathematical Society Publishing House
Seminar for Applied Mathematics
ETH-Zentrum FLI C4
CH-8092 Zürich
Switzerland

Phone: +41 (0)44 632 34 36
Email: info@ems-ph.org
Homepage: www.ems-ph.org

Printed in Germany

9 8 7 6 5 4 3 2 1

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This work is dedicated
to the memory of my father

LAMBROS CHRISTODOULOU

born Alexandria 1913
deceased Athens 1999
whose kindness is the
fondest memory I have

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Prologue and Summary

The equations describing the motion of a perfect fluid were first formulated by Euler in 1752 (see [Eu1], [Eu2]), based, in part, on the earlier work of D. Bernoulli [Be]. These equations were among the first partial differential equations to be written down, preceded, it seems, only by D'Alembert's 1749 formulation [DA] of the one-dimensional wave equation describing the motion of a vibrating string in the linear approximation. In contrast to D'Alembert's equation however, we are still, after the lapse of two and a half centuries, far from having achieved an adequate understanding of the observed phenomena which are supposed to lie within the domain of validity of the Euler equations.

The phenomena displayed in the interior of a fluid fall into two broad classes, the phenomena of sound, the linear theory of which is acoustics, and the phenomena of vortex motion. The sound phenomena depend on the compressibility of a fluid, while the vortex phenomena occur even in a regime where fluid may be considered to be incompressible. The formation of shocks, the subject of the present monograph, belongs to the class of sound phenomena, but lies in the nonlinear regime, beyond the range covered by linear acoustics. The phenomena of vortex motion include the chaotic form called turbulence, the understanding of which is one of the great challenges of science.

Let us make a short review of the history of the study of the phenomena of sound in fluids, in particular the phenomena of the formation and evolution of shocks in the nonlinear regime. At the time when the equations of fluid mechanics were first formulated, thermodynamics was in its infancy, however it was already clear that the local state of a fluid as seen by a comoving observer is determined by two thermodynamic variables, say pressure and temperature. Of these, only pressure entered the equations of motion, while the equations involve also the density of the fluid. Density was already known to be a function of pressure and temperature for a given type of fluid. However in the absence of an additional equation, the system of equations at the time of Euler, which consisted of the momentum equations together with the equation of continuity, was underdetermined, except in the incompressible limit. The additional equation was supplied by Laplace in 1816 [La] in the form of what was later to be called the adiabatic condition, and allowed him to make the first correct calculation of the speed of sound.

The first work on the formation of shocks was done by Riemann in 1858 [Ri]. Riemann considered the case of isentropic flow with plane symmetry, where the equations of fluid mechanics reduce to a system of conservation laws for two unknowns and with two independent variables, a single space coordinate and time. He introduced for such systems the so-called Riemann invariants, and with the help of these showed that solutions which arise from smooth initial conditions develop infinite gradients in finite time. Riemann also realized that the solutions can be continued further as discontinuous solutions, but here there was a problem. Up to this time the energy equation was considered to be simply a consequence of the laws of motion, not a fundamental law in its own right. On the other hand, the adiabatic condition was considered by Riemann to be part of the main framework. Now as long as the solutions remain smooth it does not matter which of the two equations we take to be the fundamental law, for each is a consequence of the other, modulo the remaining laws. However this is no longer the case once discontinuities develop, so one must make a choice as to which of the two equations to regard as fundamental and therefore remains valid thereafter.

Here Riemann made the wrong choice. For, only during the previous decade, in 1847, had the first law of thermodynamics been formulated by Helmholtz [He], based in part on the experimental work of Joule on the mechanical equivalence of heat, and the general validity of the energy principle had thereby been shown.

In 1865 the concept of entropy was introduced into theoretical physics by Clausius [Cl2], and the adiabatic condition was understood to be the requirement that the entropy of each fluid element remains constant during its evolution. The second law of thermodynamics, involving the increase of entropy in irreversible processes, had first been formulated in 1850 by Clausius [Cl1] without explicit reference to the entropy concept. After these developments the right choice in Riemann's dilemma became clear. The energy equation must remain at all times a fundamental law, but the entropy of a fluid element must jump upward when the element crosses a hypersurface of discontinuity. The formulation of the correct jump conditions that must be satisfied by the thermodynamic variables and the fluid velocity across a hypersurface of discontinuity was begun by Rankine in 1870 [Ra] and completed by Hugoniot in 1889 [Hu].

With Einstein's discovery of the special theory of relativity in 1905 [Ei], and its final formulation by Minkowski in 1908 [Mi] through the introduction of the concept of spacetime with its geometry, the domain of geometry being thereby extended to include time, a unity was revealed in physical concepts which had been hidden up to this point. In particular, the concepts of energy density, momentum density or energy flux, and stress, were unified into the concept of the energy-momentum-stress tensor and energy and momentum were likewise unified into a single concept, the energy-momentum vector. Thus, when the Euler equations were extended to become compatible with special relativity, it was obvious from the start that it made no sense to consider the momentum equations without considering also the energy equation, for these two were parts of a single tensorial law, the energy-momentum conservation law. This law together with the particle conservation law (the equation of continuity of the non-relativistic theory), constitute the laws of motion of a perfect fluid in the relativistic theory. The adiabatic condition is then a consequence for smooth solutions.

A new basic physical insight on the shock development problem was reached first, it seems, by Landau in 1944 [Ln]. This was the discovery that the condition that the entropy jump be positive as a hypersurface of discontinuity is traversed from the past to the future, should be equivalent to the condition that the flow is evolutionary, that is, that conditions in the past determine the fluid state in the future. More precisely, what was shown by Landau was that the condition of determinism is equivalent, at the linearized level, to the condition that the tangent hyperplane at a point on the hypersurface of discontinuity, is on one hand contained in the exterior of the sound cone at this point corresponding to the state before the discontinuity, while on the other hand intersects the sound cone at the same point corresponding to the state after the discontinuity, and that this latter condition is equivalent to the positivity of the entropy jump.

This is interesting from a general philosophical point of view, because it shows that irreversibility can arise, even though the laws are all time-reversible, once the solution ceases to be regular. To a given state at a given time there always corresponds a unique state at any given later time. If the evolution is regular in the associated time interval, then the reverse is also true: to a given state at a later time there corresponds a unique state at any given earlier time, the laws being time-reversible. This reverse statement is however false if there is a shock during the time interval in question. Thus determinism in the presence of hypersurfaces of discontinuity selects a direction of time and the requirement of determinism coincides, modulo the other laws, with what is dictated by the second law of thermodynamics which is in its nature irreversible. This recalls the interpretation of entropy, first discovered by Boltzmann in 1877 [Bo], as a measure of disorder at the microscopic level. An increase of entropy was thus understood to be associated to an increase in disorder or to loss of information, and determinism can only be expected in the time direction in which information is lost, not gained.

An important mathematical development with direct application to the equations of fluid mechanics in the physical case of three space dimensions, was the introduction by Friedrichs of the concept of a symmetric hyperbolic system in 1954 [F] and his development of the theory of such systems. It is through this theory that the local existence and domain of dependence property of solutions of the initial value problem associated to the equations of fluid mechanics are established. Another development in connection to this was the general investigation by Friedrichs and Lax in 1971 [F-L] (see also [Lx1]) of nonlinear first order systems of conservation laws which for smooth solutions have as a consequence an additional conservation law. This is the case for the system of conservation laws of fluid mechanics, which consists of the particle and energy-momentum conservation laws, which for smooth solutions imply the conservation law associated to the entropy current. It was then shown that if the additional conserved quantity is a convex function of the original quantities, the original system can be put into symmetric hyperbolic form. Moreover, for discontinuous solutions satisfying the jump conditions implied by the integral form of the original conservation laws, an inequality for the generalized entropy was derived.

The problem of shock formation for the equations of fluid mechanics in one space dimension, and more generally for systems of conservation laws in one space dimension, was studied by Lax in 1964 [Lx2], and 1973 [Lx3], and John [J1] in 1974. The approach of these works was analytic, the strategy being to deduce an ordinary differential inequality for a quantity constructed from the first derivatives of the solution, which showed that this quantity must blow up in finite time, under a certain structural assumption on the system called genuine nonlinearity and suitable conditions on the initial data. The genuine nonlinearity assumption is in particular satisfied by the non-relativistic compressible Euler equations in one space dimension provided that the pressure is a strictly convex function of the specific volume.

A more geometric approach in the case of systems with two unknowns was developed by Majda in 1984 [Ma1] based in part on ideas introduced by Keller and Ting in 1966 [K-T]. In this approach, which is closer in spirit to the present monograph, one considers the evolution of the inverse density of the characteristic curves of each family and shows that under appropriate conditions this inverse density must somewhere vanish within finite time. In this way, not only were the earlier blow-up results reproduced, but, more importantly, insight was gained into the nature of the breakdown. Moreover Majda's approach also covered the case where the genuine nonlinearity assumption does not hold, but we have linear degeneracy instead. He showed that in this case, global-in-time smooth solutions exist for any smooth initial data.

The problem of the global-in-time existence of solutions of the equations of fluid mechanics in one space dimension was treated by Glimm in 1965 [Gl] through an approximation scheme involving at each step the local solution of an initial value problem with piecewise constant initial data. The convergence of the approximation scheme then produced a solution in the class of functions of bounded variation. Now, by the previously established results on shock formation, a class of functions in which global existence holds must necessarily include functions with discontinuities, and the class of functions of bounded variation is the simplest class having this property. Thus, the treatment based on the total variation, the norm in this function space, in itself an admirable investigation, would be insuperable if the development of the one-dimensional theory was the goal of scientific effort in the field of fluid mechanics. However that goal can only be the mathematical description of phenomena in real three-dimensional space and one must ultimately face the fact that methods based on the total variation do not generalize to more than one space dimension. In fact it is clear from the study of the linearized theory, acoustics, which involves the wave equation, that in more than one space dimension only methods based on the energy concept are appropriate.

The first and thus far the only general result on the formation of shocks in three-dimensional fluids was obtained by Sideris in 1985 [S]. Sideris considered the compressible Euler equations in the case of a classical ideal gas with adiabatic index $\gamma > 1$ and with initial data which coincide with those of a constant state outside a ball. The assumptions of his theorem on the initial data were that there is an annular region bounded by the sphere outside which the constant state holds, and a concentric sphere in its interior, such that a certain integral in this annular region of $\rho - \rho_0$, the departure of the density ρ from its value ρ_0 in the constant state, is positive, while another integral in the same region of ρv^r, the radial momentum density, is non-negative. These integrals involve kernels which are functions of the distance from the center. It is also assumed that everywhere in the annular region the specific entropy s is not less than its value s_0 in the constant state. The conclusion of the theorem is that the maximal time interval of existence of a smooth solution is finite. The chief drawback of this theorem is that it tells us nothing about the nature of the breakdown. Also the method relies on the strict convexity of the pressure as a function of the density displayed by the equation of state of an ideal gas, and does not extend to more general equations of state.

The other important work on shocks in three space dimensions was the 1983 work of Majda [Ma2], [Ma3], on what he calls the shock front problem. In this problem we are given initial data in $\Re^3$ which is smooth in the closure of each component of $\Re^3 \setminus S$, where S is a smooth surface in $\Re^3$. The data is to satisfy the condition that there exists a function σ on S such that the jumps of the data across S satisfy the Rankine–Hugoniot jump conditions as well as the entropy condition with σ in the role of the shock speed. The higher order compatibility conditions associated to an initial boundary value problem are also required to be satisfied. We are then required to find a time interval $[0, \tau]$, a smooth hypersurface K in the spacetime slab $[0, \tau] \times \Re^3$ and a solution of the compressible Euler equations which is smooth in the closure of each component of $[0, \tau] \times \Re^3 \setminus K$, and satisfies across K the Rankine–Hugoniot jump conditions as well as the entropy condition. We may think of this problem as the local-in-time shock continuation problem. Majda solved this problem under an additional condition on the initial data which seems to be necessary for the stability of the linearized problem. The additional condition follows from the other conditions in the case of a classical ideal gas, but it does not follow for a general equation of state.

The present monograph considers the relativistic Euler equations in three space dimensions for a perfect fluid with an arbitrary equation of state. We consider regular initial data on a space-like hyperplane Σ_0 in Minkowski spacetime which outside a sphere coincide with the data corresponding to a constant state. We consider the restriction of the initial data to the exterior of a concentric sphere in Σ_0 and we consider the maximal classical development of this data. Then, under a suitable restriction on the size of the departure of the initial data from those of the constant state, we prove certain theorems which give a complete description of the maximal classical development, which we call maximal solution. In particular, the theorems give a detailed description of the geometry of the boundary of the domain of the maximal solution and a detailed analysis of the behavior of the solution at this boundary. A complete picture of shock formation in three-dimensional fluids is thereby obtained. Also, sharp sufficient conditions on the initial data for the formation of a shock in the evolution are established and sharp lower and upper bounds for the temporal extent of the domain of the maximal solution are derived.

The reason why we consider only the maximal development of the restriction of the initial data to the exterior of a sphere is in order to avoid having to treat the long time evolution of the portion of the fluid which is initially contained in the interior of this sphere. For, we have no method at present to control the long time behavior of the pointwise magnitude of the vorticity of a fluid portion, the vorticity satisfying a transport equation along the fluid flow lines. Our approach to the general problem is the following. We show that given arbitrary regular initial data which coincide with the data of a constant state outside a sphere, if the size of the initial departure from the constant state is suitably small, we can control the solution for a time interval of order $1/\eta_0$, where η_0 is the sound speed in the surrounding constant state. We then show that at the end of this interval a thick annular region has formed, bounded by concentric spheres, where the flow is irrotational and isentropic, the constant state holding outside the outer sphere. We then study the maximal classical development of the restriction of the data at this time to the exterior of the inner sphere. In the irrotational isentropic case there is a function ϕ which we call a wave function, the differential of which at a point determines the state of the fluid at that point, and the fluid equations reduce to a nonlinear wave equation for ϕ, as is shown in Chapter 1.

The order of presentation in this monograph is however the reverse of that just outlined. After the first four chapters which set up the general framework, we confine attention to the irrotational isentropic problem up to Chapter 13, where the main theorem, Theorem 13.1, is proved. We return to the general problem in Chapter 14, after establishing a theorem, Theorem 14.1, which, in the irrotational isentropic context, gives sharp sufficient conditions on the initial data for the formation of a shock in the evolution. It is at this point where our treatment of the general problem resumes, and we analyze the solution of the general problem during the initial time interval. In fact, our analysis allows us to find which conditions on the data at the beginning of the time interval result in data at the end of the time interval verifying the assumptions of Theorem 14.1. In this way we are able to establish a theorem, Theorem 14.2, which, in the general context of fluid mechanics, gives sharp sufficient conditions on the initial data for the formation of a shock in the evolution. We should emphasize at this point that if we were to restrict ourselves from the beginning to the irrotational isentropic case, we would have no problem extending the treatment to the interior region, thereby treating the maximal solution corresponding to the data on the complete initial hyperplane Σ_0. In fact, it is well known that sound waves decay in time faster in the interior region and our constructions can readily be extended to cover this region. It is only our present inability to achieve long time control of the magnitude of the vorticity along the flow lines of the fluid, that prevents us from treating the interior region in the general case.

The geometry of the boundary of the domain of the maximal solution is studied in Chapter 15, the main results being expressed by Theorem 15.1 and Propositions 15.1, 15.2, and 15.3. The boundary consists of a regular part and a singular part. Each component of the regular part $\underline{C}$ is an incoming characteristic hypersurface with a singular past boundary. The singular part of the boundary of the domain of the maximal solution is the locus of points where the inverse density of the wave fronts vanishes. It is the union $\partial_- H \bigcup H$, where each component of $\partial_- H$ is a smooth embedded surface in Minkowski spacetime, the tangent plane to which at each point is contained in the exterior of the sound cone at that point.

On the other hand each component of H is a smooth embedded hypersurface in Minkowski spacetime, the tangent hyperplane to which at each point is contained in the exterior of the sound cone at that point, with the exception of a single generator of the sound cone, which lies on the hyperplane itself. The past boundary of a component of H is the corresponding component of $\partial_- H$. The latter is at the same time the past boundary of a component of $\underline{C}$. As is explained in the Epilogue, the maximal classical solution is the physical solution of the problem up to $\underline{C} \bigcup \partial_- H$, but not up to H. The problem of the physical continuation of the solution is set up in the Epilogue as the shock development problem. This problem is associated to each component of $\partial_- H$ and its solution requires the construction of a hypersurface of discontinuity K, lying in the past of the corresponding component of H, but having the same past boundary as the latter, namely the given component of $\partial_- H$. Thus, although the notion of maximal classical solution is not physically appropriate up to H, it does provide the basis for constructing the physical continuation, the solution of the shock development problem, by providing not only the right boundary conditions at $\underline{C} \bigcup \partial_- H$, but also a barrier at H which is indispensable for controlling the physical continuation. The actual treatment of the shock development problem and the subsequent shock interactions shall be the subject of a subsequent monograph.

The present monograph concludes with a derivation of a formula for the jump in vorticity across K, which shows that while the flow is irrotational ahead of the shock, it acquires vorticity immediately behind, which is tangential to the shock front and is associated to the gradient along the shock front of the entropy jump.

We have chosen to work in this monograph with the relativistic Euler equations rather than confining ourselves to their non-relativistic limit, for three reasons. The first is the obvious reason that there is a class of natural phenomena, those of relativistic astrophysics, which lie beyond the domain of the non-relativistic equations. The second reason is that there is a substantial gain in geometric insight in considering the relativistic equations. At a fundamental level, the picture looks simpler from the relativistic perspective, because of the aforementioned unity of physical concepts brought about by the spacetime geometry viewpoint of special relativity. As an example we give the equation (1.51) of Chapter 1:

$$i_u \omega = -\theta ds. \tag{1}$$

Here ω is the vorticity 2-form. According to the definitions of Chapter 1,

$$\omega = d\beta \tag{2}$$

where β is the 1-form defined, relative to an arbitrary system of coordinates, by:

$$\beta_\mu = -\sqrt{\sigma} u_\mu, \qquad u_\mu = g_{\mu\nu} u^\nu, \tag{3}$$

$\sqrt{\sigma}$ being the relativistic enthalpy per particle, u^μ the fluid velocity and $g_{\mu\nu}$ the Minkowski metric. In (1), θ is the temperature and s the entropy per particle, while i_u denotes contraction on the left by the vectorfield u. Equation (1) is arguably the simplest explicit form of the energy-momentum equations. Our derivation in the Epilogue of the jump in vorticity behind a shock relies on this equation. The 1-form β plays a fundamental role in this monograph. In the irrotational isentropic case it is given by $\beta = d\phi$, where ϕ is the wave function.

The third reason why we have chosen to work with the relativistic equations is that no special care is needed to extract information on the non-relativistic limit. This is due to the fact that the non-relativistic limit is a regular limit, obtained by letting the speed of light in conventional units tend to infinity, while keeping the sound speed fixed. To allow the results in the non-relativistic limit to be extracted from our treatment in a straightforward manner, we have chosen to avoid summing quantities having different physical dimensions when such sums would make sense only when a unit of velocity has been chosen, even though we follow the natural choice within the framework of special relativity of setting speed of light equal to unity in writing down the relativistic equations of motion. We shall presently give an example to illustrate what we mean. Consider the vectorfield K_0 defined by equation (5.15) of Chapter 5:

$$K_0 = (\eta_0^{-1} + \alpha^{-1}\kappa)L + \underline{L}, \qquad \underline{L} = \alpha^{-1}\kappa L + 2T. \tag{4}$$

The terms here have not yet been defined, but the reader may return to this example after assimilating the appropriate definitions. In any case, η_0 is, as mentioned above, the sound speed in the surrounding constant state. The function α is the inverse density of the hyperplanes Σ_t corresponding to the constant values of the time coordinate t, with respect to the acoustical metric $h_{\mu\nu}$:

$$h_{\mu\nu} = g_{\mu\nu} + (1 - \eta^2)u_\mu u_\nu, \qquad u_\mu = g_{\mu\nu}u^\nu, \tag{5}$$

$g_{\mu\nu}$ being again the Minkowski metric, η the sound speed, and u^μ the fluid velocity. This is a Lorentzian metric on spacetime, the null cones of which are the sound cones. The function α has the physical dimension of velocity. The function κ is the inverse spatial density of the wave fronts with respect to the acoustical metric, a dimensionless quantity. Thus in the sum $\eta_0^{-1} + \alpha^{-1}\kappa$, which is the coefficient of L in the first term of (4), each term has the physical dimension of inverse velocity. The vectorfield L is the tangent vectorfield of the bicharacteristic generators, parametrized by t, of a family of outgoing characteristic hypersurfaces C_u, the level sets of an acoustical function u. The wave fronts $S_{t,u}$ are the surfaces of intersection $C_u \bigcap \Sigma_t$. The physical dimension of L is inverse time. Thus the first term in (4) has the physical dimension of inverse length. The vectorfield T defines a flow on each of the Σ_t, taking each wave front onto another wave front, the normal, relative to the induced acoustical metric $\overline{h}$, the flow of the foliation of Σ_t by the surfaces $S_{t,u}$. It has the physical dimension of an inverse length. The first term in the second part of (4) also has the same physical dimension, hence the physical dimension of the vectorfield $\underline{L}$ is inverse length as well. We conclude that each term in (4) has the physical dimension of inverse length, thus the physical dimension of K_0 is inverse length.

Denoting, as above, by σ the square of the relativistic enthalpy per particle, we have:

$$\sqrt{\sigma} = e + pv \tag{6}$$

where e is the relativistic energy per particle, p the pressure and v is the volume per particle. Let H be the function defined by:

$$1 - \eta^2 = \sigma H. \tag{7}$$

The derivative of H with respect to σ at constant s plays a central role in shock theory . This quantity is expressed by (see equation (E.47) in the Epilogue):

$$\left(\frac{dH}{d\sigma}\right)_s = -a\left\{\left(\frac{d^2v}{dp^2}\right)_s + \frac{3v}{\sqrt{\sigma}}\left(\frac{dv}{dp}\right)_s\right\} \tag{8}$$

where a is the positive function:

$$a = \frac{\eta^4}{2\sigma v^3}.$$

The sign of $(dH/d\sigma)_s$ in the state ahead of a shock determines, as is shown in the Epilogue, the sign of the jump in pressure in crossing the shock to the state behind. The jump in pressure is positive if this quantity is negative, the reverse otherwise. The value of $(dH/d\sigma)_s$ in the surrounding constant state is denoted by ℓ in this monograph. This constant determines the character of the shocks for small initial departures from the constant state. In particular when $\ell = 0$, no shocks form and the domain of the maximal classical solution is complete. Consider the function $(dH/d\sigma)_s$ as a function of the thermodynamic variables p and s. Suppose that we have an equation of state such that at some value s_0 of s we have $(dH/d\sigma)_{s=s_0} = 0$, that is, the function $(dH/d\sigma)_s$ vanishes everywhere along the adiabat $s = s_0$. We show in Chapter 1 that in this case the irrotational isentropic fluid equations corresponding to the value s_0 of the entropy are equivalent to the minimal surface equation, the wave function ϕ defining a minimal graph in a Minkowski spacetime of one more spatial dimension. Thus the minimal surface equation defines a dividing line between two different types of shock behavior.

Now, the relativistic enthalpy is dominated by the term mc^2, the contribution of the particle rest mass m, to the energy per particle, c being the speed of light. We note here that the particle rest mass may be taken to be unity, so that all quantities per particle are quantities per unit rest mass. Thus in the non-relativistic limit the second term in parenthesis in (8) vanishes and the expression in parenthesis reduces simply to: $(d^2v/dp^2)_s$.

Now, the case where $(d^2v/dp^2)_s > 0$, the adiabats being convex curves in the p, v plane, so that $(dH/d\sigma)_s < 0$, is the more commonly found in nature, however the reverse case does occur in the gaseous region near the critical point in the liquid-to-vapor phase transition and in similar transitions at higher temperatures associated to molecular dissociation and to ionization (see [Z-R]).

One of the basic concepts on which our approach relies is the general concept of variation, or variation through solutions, on which our treatment not only of the irrotational isentropic case but also of the general equations of motion is based. This concept has been discussed in the general context of Euler–Lagrange equations, that is, systems of partial differential equations arising from an action principle, in the monograph [Ch]. It was shown there that to a variation is associated a linearized Lagrangian, and it was also shown how energy currents are in general constructed on the basis of this linearized Lagrangian. It is through energy currents and their associated integral identities that the estimates essential to our approach are derived. Here the first order variations correspond to the one-parameter subgroups of the Poincaré group, the isometry group of Minkowski spacetime, extended by the one-parameter scaling or dilation group, which leave the surrounding constant state invariant. The higher order variations correspond to the one-parameter groups of diffeomorphisms generated by a set of vectorfields, the commutation fields, to be discussed below. The construction in [Ch] of an energy current requires a multiplier vectorfield which at each point belongs to the closure of the positive component of the inner characteristic core in the tangent space at that point. In the irrotational isentropic case the characteristic in the tangent space at a point consists only of the sound cone at that point and this requirement becomes the requirement that the multiplier vectorfield be non-space-like and future-directed with respect to the acoustical metric (5). We use two multiplier vectorfields in our analysis of the isentropic irrotational problem.

The first is the vectorfield K_0 defined by (4) and the second is the vectorfield K_1 defined by equation (5.16) of Chapter 5:

$$K_1 = (\omega/\nu)L. \tag{9}$$

Here ν is the mean curvature of the wave fronts $S_{t,u}$, sections of the outgoing characteristic hypersurfaces C_u, relative to their characteristic normal L, the tangent vectorfield to the bi-characteristic generators of C_u, parametrized by t. However ν is defined not relative to the acoustical metric $h_{\mu\nu}$ but rather relative to a conformally related metric $\tilde{h}_{\mu\nu}$:

$$\tilde{h}_{\mu\nu} = \Omega h_{\mu\nu}. \tag{10}$$

It turns out that there is a choice of conformal factor Ω such that in the isentropic irrotational case a first order variation $\dot{\phi}$ of the wave function ϕ satisfies the wave equation relative to the metric $\tilde{h}_{\mu\nu}$. This is shown in Chapter 1 and this choice defines Ω in the remainder of the monograph. The definition makes Ω the ratio of a function of σ to the value of this function in the surrounding constant state, thus Ω is equal to unity in the constant state. It turns out moreover that Ω is bounded above and below by positive constants. The function ω appearing in (9) is required to satisfy certain conditions (conditions **D1**–**D5** of Chapter 5) and it is shown in Proposition 13.4 that the function $\omega = 2\eta_0(1+t)$ does satisfy these requirements. A similar analysis to the one done above in the case of the multiplier field K_0 shows, taking into account the fact that the physical dimension of ν is inverse time, that the multiplier field K_1 has the physical dimension of length. The vectorfield K_1 corresponds to the generator of inverted time translations, which are proper conformal transformations of the Minkowski spacetime with its Minkowskian metric $g_{\mu\nu}$. The latter was first used by Morawetz [Mo] to study the decay of solutions of the initial boundary value problem for the classical wave equation outside an obstacle. The vectorfield K_1 is an analogue of the multiplier field of Morawetz for the acoustical spacetime which is the same underlying manifold but equipped with the acoustical metric $h_{\mu\nu}$. The energy currents associated to K_0 and K_1 are defined by equations (5.18) and (5.19) of Chapter 5, respectively. The energy current associated to K_1 contains certain additional lower order terms, defined through the function ω. Analogous terms were present in the work of Morawetz. The general structure of these terms has been investigated, in the general context of Euler–Lagrange equations, in [Ch].

To each variation ψ, of any order, there are energy currents associated to ψ and to K_0 and K_1 respectively. These currents define the energies $\mathcal{E}_0^u[\psi](t)$, $\mathcal{E}_1'^u[\psi](t)$, and fluxes $\mathcal{F}_0^t[\psi](u)$, $\mathcal{F}_1'^t[\psi](u)$. For given t and u the energies are integrals over the exterior of the surface $S_{t,u}$ in the hyperplane Σ_t, while the fluxes are integrals over the part of the outgoing characteristic hypersurface C_u between the hyperplanes Σ_0 and Σ_t. To obtain the energy and flux associated to K_1, certain integrations by parts are performed. This construction is presented in Chapter 5. The precise choice of the factor ω/ν in (9) is dictated by the need to eliminate certain error integrals which would otherwise be present. It is these energy and flux integrals, together with a spacetime integral $K[\psi](t,u)$ associated to K_1, to be discussed below, which are used to control the solution.

It is evident from the above that the means by which the solution is controlled depend on the choice of the acoustical function u, the level sets of which are the outgoing