MATHEMATICS IN NATURE

Modeling Patterns in the Natural World

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Mathematics in Nature

Dedicated to

MY MOTHER JOAN

and to the memory of

MY FATHER ALBERT,

in gratitude for the many sacrifices
they made for my education.

PREFACE

Mathematics in nature: this book grew out of a course of the same name, with the rather long subtitle "the beauty of nature as revealed by mathematics and the beauty of mathematics as revealed in nature." That course in turn grew out of an awakened awareness of both such facets of nature, even in a suburban environment, enhanced by occasional trips to national and state parks armed with binoculars and camera. I decided that it might be fun to develop a course that included *some* of the mathematics that lies behind *some* of the phenomena we encounter in the natural world around us. Such a course would have to be developed from "scratch," so to speak, because to my limited knowledge there was no such course in existence at that time, and apart from some fascinating books by Ian Stewart (one of which, *Nature's Numbers*, was a valuable supplement to the course) there was nothing that I felt was suitable as a rather free-ranging and perhaps "offbeat" textbook for such a course, at least in the way I wanted to teach it. Two books written by Pat Murphy for the Exploratorium (one written with Paul Doherty) contained some magnificient photographs of the kinds of things I wanted to describe in mathematical terms, so these books were recommended for the students also. The course was offered in two successive years, first at the junior level and finally at the senior/first-year graduate level, which was therefore presented at a more sophisticated mathematical level. I have appreciated the enthusiasm of the students in these classes—we all learned much and had a great deal of fun in the process, despite the class being held during "prime sleeping time" according to one of the participants! I was fortunate to have students in this class who ranged in ability from "bright" to "really, really bright," and I thank all of them for their active participation, questions, comments (and yes, Brandon, corrections). There was a strong temptation on my part (resisted) to fail them all so that I could see them in class the next time around. Bonnie Burke "caught the vision" for the course as I started to explain my early ideas about it; thank you for the continued encouragement, Bonnie.

I wanted to limit the topics covered to those objects that could be seen with the naked eye by anyone who takes his or her eyes outside. There are many books written on the mathematical principles behind phenomena that take place at the microscopic and submicroscopic levels, and also from planetary to galactic scales. But leaves, trees, spider webs, bubbles, waves, clouds, rainbows, . . . these are elements of the stuff we can easily see. The length scales extend roughly from 0.1 mm (the thickness of a human hair or the size of some ice crystals and diatoms) to almost 1000 km (large storm systems), a factor of about 10^{10}; the timescales of the phenomena we seek to describe range from a fraction of a second (the period of some ripples on a puddle) through the order of a day (tidal periods) to the time a tree takes to mature (perhaps thirty years in some cases), corresponding to a factor of about 10^8 or 10^9. Of course, in all likelihood we would not be around to see a sequoia tree reach maturity (!), so I have drawn the line somewhat arbitrarily at thirty years, but not in order to suggest that we stand around and "watch trees grow."

Many patterns are readily identified in nature. A visit to the zoo reminds us, no doubt unnecessarily, that tigers and zebras have striped patterns; leopards and hyenas have spotted patterns; giraffes are very blotchy (as well as very tall), while butterflies and moths may possess them all: spots, blotches, bands, and stripes. Everyone notices the wave patterns that move across oceans, lakes, ponds, and puddles, but fewer perhaps realize that waves move slowly across deserts in the form of sand dunes. In the sky brightly colored circular arcs—rainbows—beautify the sky after rain showers; from an airplane or a high peak there may sometimes be seen "glories"—small colored concentric rings surrounding the shadow of the airplane or observer—a phenomenon often called "the Specter of the Brocken" because it is frequently observed by climbers on the Brocken peak in Germany. On occasion circular and noncircular halos around and about the sun can be produced by ice crystals. If still we look up, we may sometimes observe parallel bands of cloud spreading across the sky; these may be billows or lee waves depending on the mechanism producing them—the latter are sometimes called mountain waves for this very reason. We may also see hexagon-like clouds hanging suspended from a blue ceiling: this is a manifestation of cellular convection. Patterns of light scattering are exhibited in the pre-dawn and twilight skies, at sunrise and sunset; blue sky and iridescent clouds. . . .

If we look *down* and around in a well-tended garden, we may become aware upon further investigation that arrangements of leaves, petals, seeds, and florets are intimately associated with spiral patterns, the related sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, . . . , and also with an *angle* of about 137.5° (or its complement, 222.5°). Moving on to bigger plants, we may note that the heights of trees are closely related to their diameters, following sound engineering principles. Spirals in three dimensions (helices) with interesting geometric properties plus striped patterns combine to make exquisite sea shells. Branching patterns in trees, leaves, river networks, lungs, and blood vessels exhibit similar (fractal-like) features; there is an amazing unity (without uniformity) in nature. There are also to be found some fascinating geometric properties associated with mud cracks and patterns in the bark of trees.

Although it developed out of a university course, this is not a textbook per se. It will be very useful as such, I hope, but it can also be dipped into at leisure; simple examples are scattered liberally throughout the book, and especially in the early chapters. For whom, then, is the book written? The answer is that it is for a mixture of communities, academic and otherwise. Certainly I have in mind the college population of undergraduate students in mathematics, science, and engineering (and their professors), who may be able to use it as a supplement to their standard texts in various courses, particularly those in *mathematical modeling*. The material covered here is very broad in its scope, and I hope that it will be of considerable interest to professors and students engaged in both these and interdisciplinary courses. My further hope is that this material will appeal to high school teachers and their students who may have the opportunity for "mathematical enrichment" beyond the normal syllabus, if time permits.

Anyone interested in the beauty of nature, regardless of mathematical background will also (I trust) enjoy much in this book. Although the mathematical level ranges across a broad swath, from "applied arithmetic" to partial differential equations, there is a measure of nonmathematical discussion of the basic science behind the equations that I hope will also appeal to many others who might wish to ignore the equations (but not at their peril). Thus those who have no formal mathematical background will find much of value in the descriptive material contained here.

The level of mathematics used varies from basic arithmetic, geometry, and trigonometry through calculus of a single variable (and a smidgin of linear algebra) to the *occasional* senior or first-year graduate level topic in American universities and colleges. A background in geometry, trigonometry, and single-variable calculus will suffice for most purposes; familiarity with the theory of linear ordinary and partial differential equations is useful but definitely not necessary to appreciate the contents of this book. It should therefore be accessible almost in its entirety to students of mathematics, science, and engineering, the occasional advanced topic notwithstanding.

One of the other major reasons for writing this book is to bring together different strands from the many fascinating books and scientific articles, both technical and popular, that I have collected, read and used, or just dipped into over the last twenty-five to thirty years. Some of the books are out of print, though fortunately many are not. I have been richly blessed and stimulated by the writings of many scientists and mathematicians during this time, and in one sense, therefore, this book is the result of having carried out some "intellectual janitorial activity," a phrase that I encountered in Blair Kinsman's book *Wind Waves* many years ago and have adopted as my own. My hope is that this book will be a valuable resource for you, the reader; it may provide details of previously unknown sources that I encourage you to search out for yourself if time permits, but failing that, may it be a useful introduction to some of the fascinating and varied research that has been carried out by some very clever people!

Continuing this personal theme, I would like to share my philosophy for both writing about and teaching applied mathematics. It is a simple one: (i) try to understand the material to be presented at as many levels of description as is reasonable, and (ii) attempt to communicate that understanding with enthusiasm, gentleness, and humor. Like most others in my profession, I continue to be fascinated by the beauty, power, and applicability of mathematics, and I try to induce that fascination in others (often with mixed success in the classroom). Mathematics is a subject that is misperceived, sadly, by many both inside and outside the academic world. It is thought either to involve "doing long sums" or to be a cold, austere subject of little interest in its own right and no practical application whatsoever. These extremes could not be further from the truth, and one of my goals in teaching mathematics is to try to open students' minds to the above-mentioned triad of beauty, power, and applicability (even one out of three would be valuable!). My goals are the same in writing.

Nowadays a great deal of what is taught in universities and colleges by applied mathematicians comes under the general description *mathematical modeling*. Mathematical modeling is as much "art" as "science": it requires the practitioner to (i) identify a so-called "real world" problem (whatever the context may be); (ii) formulate it in mathematical terms (the "word problem" so beloved of undergraduates); (iii) solve the problem thus formulated (if possible; perhaps approximate solutions will suffice, especially if the complete problem is intractable); and (iv) interpret the solution in the context of the original problem. (What does this answer tell me? What does it really mean? Is it consistent with what I know already about the problem? What predictions can be made?) The formulation stage is often the most difficult: it involves making judicious simplifications to "get a handle" on the salient features of the problem. This in turn provides the basis in principle for a more sophisticated model, and so on. Whether the class is at an elementary, intermediate, or advanced level, it is important to convey aspects of the modeling process that are relevant to a particular mathematical result or technique that may be discussed in that class. Often this is most easily accomplished by illustrating the result in the context of a particular application. Consequently there are plenty of applications in this book.

Stephen Wolfram's recent book *A New Kind of Science* offers a fascinating and thought-provoking alternative to classical applied mathematics. He writes: "The traditional mathematical approach to science has historically had its great success in physics—and by now it has become almost universally assumed that any serious physical theory must be based on mathematical equations . . . we will see that some extremely simple programs seem able to capture the essential mechanisms for a great many physical phenomena that have previously seemed completely mysterious. . . . Existing methods in theoretical physics tend to revolve around ideas of continuous numbers and calculus—or sometimes probability. Yet most of the systems in this book involve just simple definite discrete elements with definite rules."

The question arises: does this approach to modeling natural phenomena really explain the mechanisms underlying the observed patterns? Or do such cellular automata reproduce and describe them without explaining them? No doubt time will tell, as the implications of this challenging book are assimilated and debated both within and outside the scientific community.

Some will wonder why I have not included a chapter on "Fractals and Chaos in Nature," since fractals, chaos, and "complexity" are of such interest within the scientific community. There are two basic reasons: (i) many others have done an excellent job already on this topic, and (ii) it is a huge subject that could occupy several volumes if carried out properly. Fractal geometry has been characterized by some as the only realistic way to mimic nature and describe it in mathematical terms, and without wishing to question the foundations upon which this book is based (non-fractal mathematics), there is a measure of truth to that statement. But it has always been a strongly held opinion of mine that, in order to gain the most understanding of a physical phenomenon, it is necessary to view it with as many complementary levels of description and explanation as possible (but see p. 340 for a different perspective). The "classical applied mathematics" utilized in this book represents some of these levels, of course, and fractal geometry represents another, very profound approach. Furthermore, the latter is always "lurking in the background" in a book of this type, and to this end I have provided a very brief and cursory appendix on the topic of fractals and chaos, with some quotes from experts in the field and references for further reading. Indeed, the bibliography contains chapter numbers after each reference to indicate which chapter(s) draw on this reference, either as source material or as a valuable place to go deeper into a topic. There are many valuable sources in the literature, and I consider it a privilege if I can point readers to some of these in their quest for a better appreciation of "mathematics in nature." Yet another approach to the scientific and mathematical description of nature is via statistics and probability theory, and while those subjects do not constitute a major thread in this book, they are discussed in a little more detail in chapter 1.

A word about the epigraphs at the beginning of each chapter: they are from the New International Version (NIV) of the Old Testament, with the exceptions of those used in chapters 8, 10, and 11 which are from the King

James Version (KJV), and that in chapter 14, from the Revised Standard Version (RSV). As many theologians have said wisely, "A text without a context is a pretext." These verses lie in the intersection of two different contexts, and so I urge the interested reader to investigate both. While they may seem unusual in a book of this nature, I have chosen them because I think they have bearing on the subjects of each chapter, but sometimes you will have to think about that.

ACKNOWLEDGMENTS

I am grateful for the opportunity provided by Old Dominion University (and the Department of Mathematics and Statistics in particular) to develop the course, teach it, and write the book based on my classroom notes and literally hundreds of other sources (see the bibliography). Much of that task was accomplished during a one-semester study leave in the fall of 2001. All concerned have been very supportive of these my goals, and while it may have been merely aquiescence as a result of sheer exhaustion on their part ("Just go ahead and do it, John, and leave us in peace"), I rather think it was based on genuine and mutual collegial cooperation and respect. Mark Leslie has frequently dropped mathematical "snippets" in my mailbox or enthusiastically stopped by my office with a valuable reference, article, or "discovery." To him and all my colleagues I say thank you! Our hard-working secretaries Barbara Jeffrey and Gayle Tarkelson keep our department running smoothly, and my life would have been unnecessarily complicated without their constant and cheerful support. Bill Drewry in the Department of Civil and Environmental Engineering has imparted encouragement and wise advice to me over many a cup of coffee. I am especially grateful to Debbie Miller, who painstakingly drew the many figures in this book. There are many other friends and acquaintances who by words and emails have warmly wished me well in the writing of this book; I thank you all—you know who you are.

I am extremely appreciative of the sound advice and encouragement I have received from Vickie Kearn, senior mathematics editor at Princeton University Press. Right from the start of this project Vickie "caught the vision" I had for this book and kept me on track with her patience, enthusiasm, and humor. I have also greatly benefited from conversations (both electronic and acoustic) with her counterpart in the European PUP office, David Ireland. Thank you both. I would also like to thank Alison Anderson, who copyedited the typewritten manuscript, and Anne Reifsnyder, production editor at the Press, who gallantly dealt with my corrections at the proof stage. Frankly, it was a humbling experience to find I had made so many mathematical errors when typing the manuscript (with all of two fingers).

Perhaps it is too much to hope that I caught them all at the final proof stage. In the event that errors remain, I apologize for them and ask the reader's forbearance; they are my responsibility alone.

My wife Susan and children Rachel, Matthew, and Lindsay have been tremendously supportive and increasingly exicited as this project has evolved, and to each of them I say thank you for your continued love and encouragement.

Finally, a gentle plea of mine to the reader is as follows: *be observant.* There are many optical and fluid dynamical phenomena (particularly the latter) taking place in the everyday world around you—in the sky, clouds, rivers, lakes, oceans, puddles, faucets, sinks, coffee cups, and bathtubs. I hope that as a result of reading this book you will be better able to understand such phenomena, both mathematically and physically—it is a fascinating interdisciplinary area of applied mathematics, which richly rewards those who invest some time and effort to study it.

CREDITS

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PROLOGUE

Why I Might Never Have Written This Book

At about eleven years of age, I developed a passion for astronomy. I read everything about it I could get my hands on. My parents were very supportive, but my father, being a farmworker, had a very meager weekly income, about $30 per week at that time, so not a lot of money was available to support my astronomy habit. However, they did have a small insurance policy on my life, which they cashed in for about the equivalent of $50, as I recall. With this I bought a beautiful but somewhat dented old brass-tubed telescope: a 3-inch refractor (with an army-surplus tripod) that I have to this day. I spent many evening hours outside with it, observing the sky (most of which was cloudy—this was England, after all!). That set my career path, or so I thought. I had just started what in England we called secondary school, which spans the age range eleven to eighteen. At that time we were all streamed into different ability groups; there were three at Henley-on-Thames Grammar School, where I attended. Being something of a plodder, I was placed in the middle group and remained there for most of my years at the school. I remember very distinctly my algebra teacher asking us all what we wanted to be when we grew up. When Mr. Archibald Chanter (Arch) got to me, I said proudly, "An astronomer, Sir."

Immediately a rather worried look crossed his face, as he frowned and said, "But Adam, astronomers need to know an awful lot of maths, and you are very near the bottom of the class in this subject." Looking back, that was something of a turning point for me. I had two choices, though I didn't appreciate it at the time: to get discouraged and give up, or to use this as an opportunity to rise to the challenge and work my tail off to learn and understand the mathematics I needed to become an astronomer. Providentially, I chose the latter. I'll never know exactly what was in Arch's mind at the time, but I think he was genuinely concerned that I just didn't have the smarts to be an astronomer, even though I had the telescope!

Well, things didn't exactly change overnight. I buckled down and persevered throughout the next six or seven years, sometimes spending complete weekends working on my math(s) homework. I remember another mathematics teacher writing several years later in a report card that "by dint of sheer hard work, Adam is sometimes able to achieve far more than his brighter peers." And that was the secret: plod, plod, plod on my part and the blessing of having very supportive parents. While some of those brighter peers were groomed for Oxford and Cambridge Universities, the rest of us just applied to universities elsewhere, and, to cut a long story short, in the process of gaining my Ph.D. in theoretical astrophysics from the University of London, I became so enamored of applied mathematics that I resolved to turn my career, if possible, in the direction of becoming a lecturer (professor) in that field. I am also very grateful to a famous British amateur astronomer: Sir Patrick Moore, who is something of a cult figure in the UK, and who has hosted the television program *The Sky at Night* for over 45 years. It was he who answered my letters about becoming an astronomer, encouraged me to pursue my dream, and even invited my parents and me to visit him in Armagh, Northern Ireland, during his tenure as Director of the Armagh Planetarium. That was in 1966. Thank you, Sir Patrick! Although I am not now an astronomer as I had aspired to be all those years ago, on starry nights I still contemplate what are sometimes called the astronomer's psalms—Psalms 8 and 19. And were my father and Arch alive today, I'm sure they would appreciate the fact that in their own very different ways they helped me very, very much.

So, if I had chosen a nonscientific career, I would not have written this book. I'm thankful to have the privilege and joy of dedicating it to my mother and the memory of my father. My hope is that it will not receive the kind of review that the mathematician Mark Kac allegedly wrote—"This book fills a much-needed gap in the literature. The publishers should have opted for the gap."

Mathematics in Nature

Were the whole realm of nature mine
That were an offering far too small
Love so amazing, so divine
Demands my soul, my life, my all.

ISAAC WATTS (1674–1748)

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