Engineering Sciences Materials

SOLIDIFICATION

J. A. Dantzig and M. Rappaz

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Jonathan A. Dantzig and Michel Rappaz

Solidification is one of the oldest processes for producing complex shapes for applications ranging from art to industry, and remains as one of the most important commercial processes for many materials. Since the 1980s, numerous fundamental developments in the understanding of solidification processes and microstructure formation have come from both analytical theories and the application of computational techniques using commonly available powerful computers. This book integrates these new developments in a comprehensive volume that also presents and places them in the context of more classical theories.

The three-part text is aimed at graduate and professional engineers. The first part, Fundamentals and Macroscale Phenomena, presents the thermodynamics of solutions and then builds on that subject to motivate and describe equilibrium phase diagrams. Transport phenomena are discussed next, focusing on the issues of most importance to liquid-solid phase transformations, then moving on to describing in detail both analytical and numerical approaches to solving such problems. The second part, Microstructure, employs these fundamental concepts for the treatment of nucleation, dendritic growth, microsegregation, eutectic and peritectic solidification, and microstructure competition. This part concludes with a chapter describing the coupling of macro- and microscopic phenomena in microstructure development. The third and final part describes various types of Defects that may occur, with emphasis on porosity, hot tearing and macrosegregation, presented using the modeling tools and microstructure descriptions developed earlier.

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SOLIDIFICATION

PREFACE

The modern science of solidification began in the 1940's, when engineers began to use analytical methods and models to describe solidification processes. In 1940, Chvorinov applied the analysis of heat flow to predict solidification patterns and defect in sand castings. In the 1950's Chalmers and co-workers analyzed the heat and solute balance at the moving solid-liquid interface to understand why planar interfaces become unstable during unidirectional growth. This body of work culminated in a seminal text, *Principles of Solidification*, written by Chalmers in 1964.

In the 1960's, Mullins and Sekerka put Chalmers' analysis on a firmer mathematical footing by performing a formal stability analysis. Later in the 1960's and in the 1970's, Flemings and co-workers developed models for segregation and other microstructural features by applying heat and solute balances at the scale of the microstructural features themselves. Flemings followed Chalmers' text with *Solidification Processing* in 1974, presenting this next generation of achievements.

The next decade saw a great deal of activity in the study of microstructure as a pattern selection problem through the competition between the transport of heat and solute and inherent length scales in the material owing to surface energy. This body of work was summarized in 1984 in *Fundamentals of Solidification*, by Kurz and Fisher.

Kurz and Fisher's book appeared just at the beginning of a revolution in modeling of solidification, when low-cost powerful computers became available. Computational approaches allowed more accurate and detailed models to be constructed, shedding light on many important phenomena. Today, industrial users regularly model the solidification of geometrically complex parts ranging from directionally solidified turbine blades to automotive engines. At the microscopic scale, computational models have been used to great effect to understand the pattern selection process in ways that were only hinted at using the analytical techniques available earlier. In the 1990's, methods were developed to combine these microscopic and macroscopic views of solidification processes.

Although there have been a few specialty texts written in the intervening time between Kurz and Fisher's book and the present, none of them provides a comprehensive presentation of the fundamentals, analytical models, and computational approaches. Also in the 1990's and 2000's, a short course on solidification was developed in collaboration with

Ecole des Mines de Nancy, EPFL and Calcom, that incorporated both the fundamental aspects described earlier and the developing computational techniques. Through teaching this course, and at our respective universities, we felt that there was a need for a new text, which led us to write the book you now hold in your hands. The subject is presented in three parts: *Fundamentals*, which provides the basics of thermodynamics, phase diagrams, and modeling techniques. *Microstructure* then uses these techniques to describe the evolution of the solid at the microscopic scale, from nucleation to dendrites, eutectics and peritectics to microsegregation. This section concludes with a chapter on coupling macro- and micro-models of solidification. The final part, *Defects*, uses the same principles to describe porosity, hot tearing and macrosegregation. We have striven to present this wide range of topics in a comprehensive way, and in particular to use consistent notation throughout.

Acknowledgment

This work represents the culmination of our education, training and practice over the last 25 years. We have had many mentors, colleagues and friends who have helped us along the way, too numerous to name them all. We would particularly like to express our gratitude to our mentors, Wilfried Kurz, Stephen Davis and Robert Pond, Sr. In addition to the authors mentioned above, we would like to acknowledge the very fruitful discussions we have had over the years with many esteemed colleagues: William Boettinger, Martin Glicksman, John Hunt, Alain Karma and Rohit Trivedi, to name just a few. Much of the structure of this text derived from the short course described above, and we would like to thank our colleagues and fellow teachers in that course, past and present, in particular Philippe Thévoz and Marco Gremaud who organize the course. Special thanks are due to Christoph Beckermann, Hervé Combeau, Arne Dahle and Mathis Plapp for their helpful comments and contributions to the manuscript, and to Sébastien Rappaz for the design of the cover. We also owe a special debt of gratitude to our students, postdocs and coworkers, whose contributions made this book possible. We would also like to thank our many colleagues and friends who graciously allowed us to use figures and movie sequences that appear throughout the book.

Most importantly, we would like to thank our families for their love and support.

Jonathan A. Dantzig
Michel Rappaz
Lausanne, 2009

NOMENCLATURE AND DIMENSIONLESS GROUPS

Principal Nomenclature

A book that covers as many topics as this one does is bound to encounter some problems with nomenclature. We have tried to use standard notation wherever possible, and to be consistent in usage throughout. In order to help distinguish between dimensional quantities and their dimensionless counterparts after scaling, we use Roman alphabet symbols for dimensional variables and corresponding Greek letters for the dimensionless ones. For example, the dimensional coordinates $(x, y, z) \rightarrow (\xi, \eta, \zeta)$. For some variables, such as velocity $\boldsymbol{v}, v_i$, this scheme is not possible because there is no Greek counterpart. Further, we need symbols for both the vector and its components. We handle this by using italic symbols for dimensional quantities and Roman symbols for the dimensionless ones, e.g., $(\boldsymbol{v}, v_i) \rightarrow (\mathbf{v}, \mathrm{v}_i)$.

Subscripts and superscripts can be complicated as well. We use upper case Roman letters to designate components, lower case Greek to represent phases, s or ℓ to designate solid and liquid, respectively, and a superscript '*' to designate quantities evaluated at the solid-liquid interface. Whereas the '*' will always appear as a superscript, the other indices may appear as subscripts or superscripts, depending on what form provides the clearest description in the current context. As an example, the most complicated symbol used in the text is $C^{*\alpha}_{J\ell}$, which means the mass fraction of component J in the liquid, evaluated at the solid-liquid interface ahead of phase α. This symbol appears in the discussion of the solidification of eutectic and peritectic alloys. The most important symbols are given below.

Roman alphabet

A, B, ...	species (component) A
A, $A_{s\ell}$	area, solid-liquid interfacial area
$A_{f\ell}$, A_{fs}	surface area between foreign substrate and liquid, or foreign substrate and solid
A_C, A_R	growth constants for eutectics
$A(\boldsymbol{n})$	surface energy anisotropy function

$a_1, a_2, a_3, \ldots$	surface energy anisotropy coefficients
$a_{A\alpha}$	chemical activity of species A in phase α
B	ratio of solutal and thermal expansion coefficients ($= \beta_C/(m_\ell \beta_T)$)
$[\boldsymbol{B}^e]$	spatial derivatives of element shape functions
$\boldsymbol{b}$	vector of body force per unit mass; design vector
$[\boldsymbol{C}^e], [\boldsymbol{C}]$	element and global capacitance matrices in FEM
C	mass fraction of solute in a binary alloy
C_J	mass fraction of species J in a mixture
C_s^*, C_ℓ^*	mass fractions of solute in the solid and liquid phases of a binary alloy at the solid-liquid interface
$c_0, c_1, \ldots$	constants of integration
c_p, c_V	specific heat at constant pressure; at constant volume
D	chemical diffusivity of solute; diameter
d	diameter of a sphere
$\boldsymbol{D}$	rate-of-deformation tensor ($= (\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)/2$)
d_0, d_0^C	thermal capillary length; chemical capillary length
E	Young's modulus
E, E^m, e	total, molar and specific internal energy
$\dot{E}$	cumulative average deformation rate
$\boldsymbol{\mathcal{E}}, \mathcal{E}_{ijkl}$	elasticity tensor, indecial form
e_J^I	second-order interaction coefficient between solute element I and gas element J
$\mathcal{F}, F$	total and volumetric free energy in phase-field model
$\boldsymbol{F}$	deformation gradient tensor, $F_{ij} = \partial x_i / \partial X_j$
f_A, f_V	geometric factors for nucleation in a conical pit
$f_{J\ell}, f_{J\ell}^0$	activity coefficient for species J in an alloy and in a pure material
f_α	mass fraction of phase α
G	temperature gradient
G, G^m, g	total, molar and specific Gibbs free energy
G_C	composition gradient
g	acceleration due to gravity, 9.82 m s^{-2}
g_α	volume fraction of phase α
g_d, g_e, g_g	volume fraction of interdendritic liquid, extradendritic liquid and grain
g_s	volume fraction of solid
g_{se}	extended volume fraction of the solid phase
g_{si}	internal volume fraction of the solid phase in a grain
H, H^m, h	total, molar and specific enthalpy
h_T	heat transfer coefficient
$\boldsymbol{I}$	unit tensor (identity tensor); the ij component is δ_{ij}
I^{homo}, I^{heter}	homogeneous or heterogeneous nucleation rate
I_0^{homo}, I_0^{heter}	prefactors for homogeneous or heterogeneous nucleation rate
$\mathrm{Iv}_{2D}, \mathrm{Iv}_{3D}$	Ivantsov function in 2-D or 3-D
i	$\sqrt{-1}$
$\boldsymbol{j}_A$	mass fraction flux for species A
J	Jacobian ($\det \boldsymbol{F}$)
$[\boldsymbol{J}]$	element Jacobian for isoparametric FEM

$\boldsymbol{K}, K$	permeability tensor; value of isotropic permeability
$[\boldsymbol{K}^e], [\boldsymbol{K}]$	element and global conductance matrices in FEM
$\boldsymbol{k}, k$	thermal conductivity tensor; value of isotropic conductivity
k_T	thermal conductivity ratio ($= k_s/k_\ell$)
k_0, k_0^m	partition coefficient (mass); partition coefficient (molar)
k_B	Boltzmann's constant, 1.38×10^{-23} J K^{-1}
L, L_c	characteristic length
L_f	latent heat of fusion per unit mass
L_v	latent heat of vaporization per unit mass
M	mass of a system; morphological number
$\mathcal{M}_J$	molecular weight of species J
m_ℓ, m_s	slopes of the liquidus and solidus curve (mass fractions)
$[\boldsymbol{N}^e]$	element shape functions in FEM
N_0	Avagadro's number, 6.02×10^{23} atoms mol^{-1}
N_C	number of components
N_F	number of degrees of freedom for phase equilibria
N_I	total number of atoms of species I in a mixture
N_b	bond coordination number
N_g	number of grid points in a computational domain
N_ϕ	number of phases
n	number of moles
$\boldsymbol{n}, (n_x, n_y, n_z)$	unit vector normal to a surface and its Cartesian components
n, n_g	density of grains
n_{max}	maximum density of particles available
n_p	density of potent nucleant particles; density of pores
$\mathcal{O}(\cdot)$	order of magnitude
P	power input to a system; penalty parameter
$P(r; R_{tip})$	surface of a paraboloid of revolution (dimensional)
$\mathcal{P}(\varrho)$	surface of a paraboloid of revolution (dimensionless)
p, p_a	pressure, atmospheric pressure
p_c	probability of capture
$p(\phi)$	orientation distribution function
$\tilde{p}$, p'	intermediate pressure and pressure correction in SIMPLE algorithm
$\hat{p}$	modified pressure ($= p + \rho_0 g h$)
Q	heat input to a system
q_b	boundary heat flux
$\boldsymbol{q}$	heat flux vector
R	radius
$R_g; R_{g0}$	radius of grain; final grain radius
R_1, R_2	principal radii of curvature
$\mathcal{R}$	gas constant, 8.31 J mol^{-1} K^{-1}
$\mathfrak{R}$	dimensionless radius for a spherical solid particle
$\dot{R}_q$	specific heat generation rate
R_c	radius of a critical nucleus
R_p	pore radius
R_{tip}	tip radius of a paraboloid
$\{\boldsymbol{R}\}$	residual vector
r, θ, z	cylindrical coordinates

r, θ, ϕ	spherical coordinates
r_J^I	second-order interaction coefficient between solute element I and gas element J
S	bounding surface
S, S^m, s	total, molar and specific entropy
S_{mix}^m	molar entropy of mixing
$\mathcal{S}_V, \mathcal{S}_V^{s\ell}$	solid-liquid interfacial area per unit volume
$\mathcal{S}_V^{de}$	interfacial area per unit volume for inter-extradendritic liquid
$\mathcal{S}_V^{sd}$	interfacial area per unit volume for solid-interdendritic liquid
T	temperature
$\dot{T}$	cooling rate
T_0	boundary temperature; temperature where $G_s^m = G_\ell^m$
T^*	solid-liquid interface temperature
T_b	boundary temperature
T_{col}	temperature of a columnar front
T_{eut}	eutectic temperature
T_f	equilibrium melting temperature of pure material
T_{liq}	liquidus temperature
T_{per}	peritectic temperature
T_{ref}, T_0	reference temperature
T_{sol}	solidus temperature
T_v	vaporization temperature at atmospheric pressure
$\{\boldsymbol{T}^e\}, \{\boldsymbol{T}\}$	local and global vector of nodal temperatures
t, t_c	time, characteristic time
t_f	local solidification time
t_n	time of nucleation
$\boldsymbol{t}$	surface traction vector
$\boldsymbol{u}$	displacement vector
V, V^m, v	total, molar and specific volume
V_R	volume of representative volume element
V_s, V_ℓ	volume of solid and liquid phases in representative volume element
v	scalar velocity
v_g	velocity normal to the surface of a grain
$\boldsymbol{v}, v_i$	(dimensional) velocity vector and its i^{th} component
$\mathbf{v}, \mathrm{v}_i$	(dimensionless) velocity vector and its i^{th} component
$\mathbf{v}_K$	velocity vector for species K
v_n	normal component of velocity of the solid-liquid interface
v_{sound}	speed of sound
v_T	isotherm velocity
W	width; total work done by external forces
W_0	phase-field interface width
X_I	molar composition of species I
$\boldsymbol{X}$	material coordinate vector
$\{\mathbf{X}\}, \{\mathbf{Y}\}, \{\mathbf{Z}\}$	element vectors of nodal coordinates in FEM
$\boldsymbol{x}$	position vector
x^*	interface position in 1-D problems
x, y, z	Cartesian coordinates; also x_1, x_2, x_3
$\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$	unit vectors in Cartesian coordinates

Greek alphabet

α	thermal diffusivity ($= k/(\rho c_p)$)
α, β, γ	generic phases
α_T	linear thermal expansion coefficient ($= \beta_T/3$)
β	Solidification shrinkage ($= \rho_s/\rho_\ell - 1$)
β_T	volumetric thermal expansion coefficient ($= 3\alpha_T$)
β_C	volumetric solutal expansion coefficient
β_p	coefficient of compressibility
$\Gamma_{s\ell}$	Gibbs-Thomson coefficient ($= \gamma_{s\ell}T_f/(\rho_s L_f)$)
$\Gamma^{m*}_{s,\ell}$, $\mathbf{\Gamma}^{\sigma*}_{s,\ell}$, $\Gamma^{h*}_{s,\ell}$, $\Gamma^{C*}_{s,\ell}$	interfacial mass, momentum, energy or species term for solid or liquid
$\gamma_{f\ell}$	surface energy between foreign substrate and liquid
γ_{fs}	surface energy between foreign substrate and solid
γ_{gb}	grain boundary energy
$\gamma_{s\ell}, \gamma^0_{s\ell}$	surface energy between solid and liquid; value of isotropic surface energy
Δ	dimensionless undercooling $c_p\Delta T/L_f$ (Stefan number)
ΔC_0	difference in compositions across eutectic plateau
$\Delta G^{homo}_n, \Delta G^{hetero}_n$	free energy barrier for homogeneous or heterogeneous nucleation
ΔH^m_{mix}	molar enthalpy of mixing
ΔS^m_f	molar entropy difference between solid and liquid
Δs^J_f	specific entropy of fusion of species J ($= L^J_f/T^J_f$)
ΔT	total undercooling
ΔT_b	undercooling for bridging or coalescence
ΔT_c	characteristic temperature difference
ΔT_0	Equilibrium freezing range ($= T_{liq} - T_{sol}$)
ΔT_k	kinetic undercooling
ΔT_n	nucleation undercooling
ΔT_R	curvature undercooling
ΔT_C	solutal undercooling
ΔT_T	thermal undercooling
$\Delta x, \Delta y, \Delta z$	grid spacing in various coordinate directions
δ	dimensionless solidified layer thickness; boundary layer thickness
ε_{JK}	bond energy between atoms of J and K
$\boldsymbol{\varepsilon}$	strain tensor
ε_{eq}	equivalent strain
$\varepsilon_4, \varepsilon_n$	4-fold, n-fold coefficient for the planar anisotropy of $\gamma_{s\ell}$
η	dimensionless $y-$coordinate; paraboloidal coordinate
ζ	dimensionless $z-$coordinate; fractional time step
θ	dimensionless temperature; angular coordinate; wetting angle
$\bar{\kappa}, \kappa_G$	mean and Gaussian curvature of a surface
Λ	ratio of eutectic spacing to extremum value ($= \lambda/\lambda_{ext}$)
λ	wavelength of instability; eutectic spacing
λ_1, λ_2	primary, secondary dendrite arm spacing
μ_ℓ	shear viscosity of a Newtonian fluid
$\mu_{J\alpha}$	chemical potential of species J in phase α

μ_k	kinetic attachment coefficient
ν_ℓ	kinematic viscosity ($= \mu_\ell/\rho_\ell$)
ν_0	atomic vibration frequency
ν_e	Poisson's ratio
ξ	dimensionless $x-$coordinate; parabolic coordinate
$\boldsymbol{\xi}$	Cahn-Hoffmann vector ($= \nabla(r\gamma_{s\ell}(\boldsymbol{n}))$)
π	3.14159...
Π	dimensionless scaled pressure
ρ	density
ρ_0	density at reference temperature and pressure
ϱ	dimensionless radial coordinate
$\boldsymbol{\sigma}$	total stress tensor
$\hat{\boldsymbol{\sigma}}$	effective stress tensor ($= \boldsymbol{\sigma} + p\boldsymbol{I}$)
σ_{eq}	equivalent stress
σ^*	dendrite tip selection constant
σ_n	instability growth rate exponent for mode n
σ_y	yield stress
$\boldsymbol{\tau}$	extra stress tensor
τ	dimensionless time
τ_0	time scale factor in phase-field model
Υ	noise in phase-field equation
ϕ	constant used to describe interface position
ϕ_s, ϕ_ℓ	existence function for solid, liquid phase
χ_α	mole fraction of phase α
ψ	phase-field order parameter
Ψ	surface stiffness
Ω^m	regular solution parameter
Ω	supersaturation
$\boldsymbol{\omega}$	vorticity vector ($= \nabla \times \boldsymbol{v}$)

Subscripts, superscripts and indices

A^*	evaluated on the solid-liquid interface
A_C	composition
A_c	characteristic value
A_{col}	columnar zone
A^{el}	elastic deformation
A_{eut}	eutectic
A_ℓ	liquid phase
A_g	gas phase
A_k	attachment kinetics
A_I, A_J	species I, species J
A_{liq}	liquidus
A^m	amount per mole
A_n	component of vector A normal to the interface
A_p	pores
A^R	surface with radius of curvature R
A_s	solid phase
A_{sol}	solidus
$A_{s\ell}$	solid-liquid interface

A^{th}	thermal deformation
A^{tr}	transformational deformation
A^{vp}	viscoplastic deformation
$A_\alpha, A_\beta, \ldots$	quantity in phase α, β, ...
A_x, A_y, A_z	x, y, z components of a vector
A_0	nominal or reference value
A^∞	flat surface ($R \to \infty$)

Mathematical functions

Symbol	Meaning	Representation
$\mathrm{E}_1(u)$	exponential integral	$\int_u^\infty \frac{e^{-s}}{s} ds$
$\mathrm{erf}\,(u)$	error function	$\frac{2}{\sqrt{\pi}} \int_0^u e^{-s^2} ds$
$\mathrm{erfc}\,(u)$	complementary error function	$1 - \mathrm{erf}\,(u)$
$f(\theta)$	nucleation geometric factor	$\frac{(2+\cos\theta)(1-\cos\theta)^2}{4}$
$\mathcal{L}_n(x)$	Laguerre polynomial	$\frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right)$
$P_{nm}(x)$	associated Legendre polynomial	$\frac{(-1)^m}{2^n n!}\left(1-x^2\right)^{m/2}\frac{d^{n+m}}{dx^{n+m}}\left(x^2-1\right)^n$
Q_4	first cubic harmonic function	$n_x^4 + n_y^4 + n_z^4$
S_4	second cubic harmonic function	$n_x^2 n_y^2 n_z^2$
$Y_{nm}(\theta,\phi)$	spherical harmonic function	$\sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} e^{-im\phi} P_{nm}(\cos\theta)$
$\delta(x)$	Dirac δ-function	$\delta(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0 \end{cases}$ $\int_{-\infty}^{\infty} \delta(x) dx = 1$
δ_{ij}	unit tensor (Kronecker delta)	$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
ε_{ijk}	permutation tensor	$\begin{cases} 1 & i,j,k \text{ even permutations} \\ -1 & i,j,k \text{ odd permutations} \\ 0 & \text{otherwise} \end{cases}$

Mathematical operators

Symbol	Meaning	Representation
$\boldsymbol{A} \cdot \boldsymbol{B}$	dot product of two vectors	$a_i b_i$
$\boldsymbol{A}^T$	transpose of a second rank tensor	a_{ji}
$\boldsymbol{A} : \boldsymbol{B}$	scalar product of second rank tensors	$a_{ij} b_{ji}$

$D\psi/Dt$	material derivative of ψ	$\frac{\partial \psi}{\partial t} + (\boldsymbol{v} \cdot \nabla)\psi$
$\mathrm{tr}\boldsymbol{A}$	trace of a second rank tensor	a_{ii}
∇A	gradient of a scalar	$\frac{\partial A}{\partial x_i}$
$\nabla \cdot \boldsymbol{A}$	divergence of $\boldsymbol{A}$	$\frac{\partial a_i}{\partial x_i}$
$\nabla \times \boldsymbol{A}$	curl of a vector	$\varepsilon_{ijk} \frac{\partial a_j}{\partial x_k}$
$\nabla^2 A$	Laplacian of A	$\frac{\partial^2 A}{\partial x_i \partial x_i}$
$\lVert \boldsymbol{A} \rVert$	L_2 norm of a vector	$\sqrt{a_i a_i}$
$\langle A \rangle$	volume average of A	$\frac{1}{V_R} \int_{V_R} A \, dV$
$\langle A_{s,\ell} \rangle$	phase average of A_s or A_ℓ	$\frac{1}{V_R} \int_{V_R} \phi_{s,\ell} A \, dV$
$\langle A \rangle_{s,\ell}$	intrinsic average of A_s or A_ℓ	$\frac{1}{V_{s,\ell}} \int_{V_R} \phi_{s,\ell} A \, dV$
$\langle A^*_{s,\ell} \boldsymbol{n} \rangle^*$	interfacial average of A_s or A_ℓ	$\frac{1}{A_{s\ell}} \int_{A_{s\ell}} A^*_{s,\ell} \boldsymbol{n} dA$
$\langle C \rangle_M$	mass average composition	$\int_0^{f_s} C_s df_s + \int_0^{f_\ell} C_\ell df_\ell$

Classical dimensionless numbers

Name	Expression	Physical Meaning
Biot	$\mathrm{Bi} = \frac{h_T L_c}{k}$	ratio of heat advection from a surface to heat conduction inside
Boussinesq	$\mathrm{Bo} = \frac{g\beta_T \Delta T_c L_c^3}{\alpha_0^2}$	ratio of heat advected by buoyancy to conducted heat
Fourier	$\mathrm{Fo} = \frac{\alpha t_c}{L_c^2}$	ratio of characteristic time t_c to the time for conduction L_c^2/α
Grashof	$\mathrm{Gr} = \frac{g\beta_T \Delta T_c L_c^3}{(\mu_\ell/\rho_{\ell 0})^2}$	ratio of buoyant advective flow to viscosity
Lewis	$\mathrm{Le} = \frac{\alpha}{D}$	ratio of thermal diffusion to mass diffusion
Péclet	$\mathrm{Pe} = \frac{v_c L_c}{\alpha}$	ratio of heat advection to heat conduction
Péclet (solutal)	$\mathrm{Pe}_C = \frac{v_c L_c}{D}$	ratio of solute advection to solute diffusion
Prandtl	$\mathrm{Pr} = \frac{c_p \mu_\ell}{k} = \frac{\nu_\ell}{\alpha}$	ratio of momentum and thermal diffusivities in a fluid

Rayleigh	$\mathrm{Ra} = \dfrac{\rho_0 g \beta_T \Delta T_c L_c^3}{\mu_\ell \alpha_0}$	ratio of buoyant advection to the product of viscosity and heat conduction
Reynolds	$\mathrm{Re} = \dfrac{\rho v_c L_c}{\mu_\ell} = \dfrac{v_c L_c}{\nu_\ell}$	ratio of inertia to viscosity
Schmidt	$\mathrm{Sc} = \dfrac{\mu_\ell}{\rho_\ell D_\ell} = \dfrac{\nu_\ell}{D_\ell}$	ratio of momentum diffusivity to mass diffusivity
Stefan	$\mathrm{Ste} = \dfrac{c_p \Delta T}{L_f}$	ratio of sensible heat to latent heat

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