An Informal Introduction to Stochastic Calculus with Applications

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NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Calin, Ovidiu.

An informal introduction to stochastic calculus with applications / by Ovidiu Calin (Eastern Michigan University, USA).

pages cm

Includes bibliographical references and index.

ISBN 978-9814678933 (hardcover : alk. paper) -- ISBN 978-9814689915 (pbk : alk. paper)

1. Stochastic analysis. 2. Calculus. I. Title. II. Title: Introduction to stochastic calculus with applications.

QA274.2.C35 2015

519.2'2--dc23

2015014680

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

Printed in Singapore by Mainland Press Pte Ltd.

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Preface

Deterministic Calculus has been proved extremely useful in the last few hundred years for describing the dynamics laws for macro-objects, such as planets, projectiles, bullets, etc. However, at the micro-scale, the picture looks completely different, since at this level the classical laws of Newtonian mechanics cease to function "normally". Micro-particles behave differently, in the sense that their state cannot be determined accurately as in the case of macro-objects; their position or velocity can be described using probability densities rather than exact deterministic variables. Consequently, the study of nature at the micro-scale level has to be done with the help of a special tool, called Stochastic Calculus. The fact that nature at a small scale has a non-deterministic character makes Stochastic Calculus a useful and important tool for the study of Quantum Mechanics.

In fact, all branches of science involving random functions can be approached by Stochastic Calculus. These include, but they are not limited to, signal processing, noise filtering, stochastic control, optimal stopping, electrical circuits, financial markets, molecular chemistry, population evolution, etc.

However, all these applications assume a strong mathematical background, which takes a long time to develop. Stochastic Calculus is not an easy theory to grasp and, in general, requires acquaintance with probability, analysis and measure theory. This fact makes Stochastic Calculus almost always absent from the undergraduate curriculum. However, many other subjects studied at this level, such as biology, chemistry, economics, or electrical circuits, might be more completely understood if a minimum knowledge of Stochastic Calculus is assumed.

The attribute *informal*, present in the title of the book, refers to the fact that the approach is at an introductory level and not at its maximum mathematical detail. Many proofs are just sketched, or done "naively" without putting the reader through a theory with all the bells and whistles.

The goal of this work is to informally introduce elementary Stochastic Calculus to senior undergraduate students in Mathematics, Economics and Business majors. The author's goal was to capture as much as possible of the spirit of elementary Calculus, which the students have already been exposed to in the beginning of their majors. This assumes a presentation that mimics similar properties of deterministic Calculus as much as possible, which facilitates the understanding of more complicated concepts of Stochastic Calculus.

The reader of this text will get the idea that deterministic Calculus is just a particular case of Stochastic Calculus and that Ito's integral is not a too much harder concept than the Riemannian integral, while solving stochastic differential equations follows relatively similar steps as solving ordinary differential equations. Moreover, modeling real life phenomena with Stochastic Calculus rather than with deterministic Calculus brings more light, detail and significance to the picture.

The book can be used as a text for a one semester course in stochastic calculus and probabilities, or as an accompanying text for courses in other areas such as finance, economics, chemistry, physics, or engineering.

Since deterministic Calculus books usually start with a brief presentation of elementary functions, and then continue with limits, and other properties of functions, we employed here a similar approach, starting with elementary stochastic processes, different types of limits and pursuing with properties of stochastic processes. The chapters regarding differentiation and integration follow the same pattern. For instance, there is a product rule, a chain-type rule and an integration by parts in Stochastic Calculus, which are modifications of the well-known rules from elementary Calculus.

In order to make the book available to a wider audience, we sacrificed rigor and completeness for clarity and simplicity, emphasizing mainly on examples and exercises. Most of the time we assumed maximal regularity conditions for which the computations hold and the statements are valid. Many complicated proofs can be skipped at the first reading without affecting later understanding. This will be found attractive by both Business and Economics students, who might get lost otherwise in a very profound mathematical textbook where the forest's scenery is obscured by the sight of the trees. A flow chart indicating the possible order the reader can follow can be found at the end of this preface.

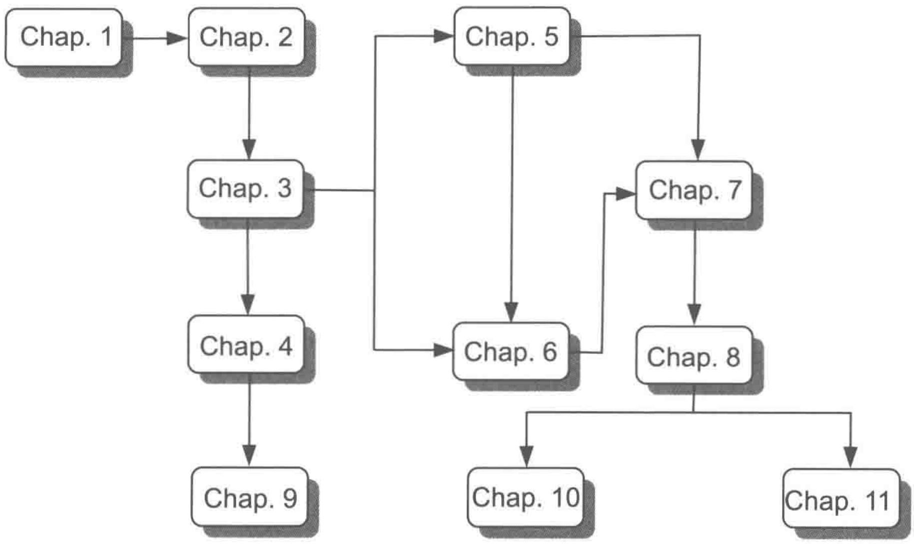
An important feature of this textbook is the large number of solved problems and examples which will benefit both the beginner as well as the advanced student.

This book grew from a series of lectures and courses given by the author at Eastern Michigan University (USA), Kuwait University (Kuwait) and Fu-Jen University (Taiwan). The student body was very varied. I had math, statistics, computer science, economics and business majors. At the initial stage, several students read the first draft of these notes and provided valuable feedback, supplying a list of corrections, which is far from exhaustive. Finding any typos or making comments regarding the present material are welcome.

Heartfelt thanks go to the reviewers who made numerous comments and observations contributing to the quality of this book, and whose time is very much appreciated.

Finally, I would like to express my gratitude to the World Scientific Publishing team, especially Rok-Ting Tan and Ying-Oi Chiew for making this endeavor possible.

O. Calin Michigan, January 2015



List of Notations and Symbols

The following notations have been frequently used in the text.

$(\Omega, \mathcal{F}, P)$	Probability space
Ω	Sample space
$\mathcal{F}$	σ-field
X	Random variable
X_t	Stochastic process
$\text{as} - \lim_{t\to\infty} X_t$	The almost sure limit of X_t
$\text{ms} - \lim_{t\to\infty} X_t$	The mean square limit of X_t
$\text{p} - \lim_{t\to\infty} X_t$	The limit in probability of X_t
$\mathcal{F}_t$	Filtration
$\mathcal{N}_t, \dot{W}_t, \frac{dW_t}{dt}$	White noise
W_t, B_t	Brownian motion
$\Delta W_t, \Delta B_t$	Jumps of the Brownian motion during time interval Δt
dW_t, dB_t	Infinitesimal jumps of the Brownian motion
$V(X_t)$	Total variation of X_t
$V^{(2)}(X_t), \langle X, X\rangle_t$	Quadratic variation of X_t
$F_X(x)$	Probability distribution function of X
$p_X(x)$	Probability density function of X
$p(x, y; t)$	Transition density function
$\mathbb{E}[\,\cdot\,]$	Expectation operator
$\mathbb{E}[X\vert\mathcal{G}]$	Conditional expectation of X with respect to $\mathcal{G}$
$Var(X)$	Variance of the random variable X
$Cov(X, Y)$	Covariance of X and Y
$\rho(X, Y), Corr(X, Y)$	Correlation of X and Y
$\mathcal{A}_X, \mathcal{F}^X$	σ-algebras generated by X

$\Gamma(\cdot)$	Gamma function
$B(\cdot,\cdot)$	Beta function
N_t	Poisson process
S_n	Waiting time for Poisson process
T_n	Interarrival time for Poisson process
$\tau_1 \wedge \tau_2$	The minimum between τ_1 and τ_2 $(= \min\{\tau_1, \tau_2\})$
$\tau_1 \vee \tau_2$	The maximum between τ_1 and τ_2 $(= \max\{\tau_1, \tau_2\})$
$\bar{\tau}_n$	Sequence superior limit $(= \sup_{n\geq 1} \tau_n)$
$\underline{\tau}_n$	Sequence inferior limit $(= \inf_{n\geq 1} \tau_n)$
μ	Drift rate
σ	Volatility, standard deviation
$\partial_{x_k}, \dfrac{\partial}{\partial x_k}$	Partial derivative with respect to x_k
$\mathbb{R}^n$	n-dimensional Euclidean space
$\|x\|$	Euclidean norm $(= \sqrt{x_1^2 + \cdots + x_n^2})$
Δf	Laplacian of f
$1_A, \chi_A$	The characteristic function of A
$\|f\|_{L^2}$	The L^2-norm $(= \sqrt{\int_a^b f(t)^2\, dt})$
$L^2[0,T]$	Squared integrable functions on $[0,T]$
$C^2(\mathbb{R}^n)$	Functions twice differentiable with second derivative continuous
$C_0^2(\mathbb{R}^n)$	Functions with compact support of class C^2
R_t	Bessel process
$\widehat{\zeta}_t$	The mean square estimator of ζ_t

Contents

Chapter 1

A Few Introductory Problems

Even if deterministic Calculus is an excellent tool for modeling real life problems, however, when it comes to random exterior influences, Stochastic Calculus is the one which can allow for a more accurate modeling of the problem. In real life applications, involving trajectories, measurements, noisy signals, etc., the effects of many unpredictable factors can be averaged out, via the Central Limit Theorem, as a normal random variable. This is related to the Brownian motion, which was introduced to model the irregular movements of pollen grains in a liquid.

In the following we shall discuss a few problems involving random perturbations, which serve as motivation for the study of the Stochastic Calculus introduced in next chapters. We shall come back to some of these problems and solve them partially or completely in Chapter 11.

1.1 Stochastic Population Growth Models

Exponential growth model Let $P(t)$ denote the population at time t. In the time interval Δt the population increases by the amount $\Delta P(t) = P(t + \Delta t) - P(t)$. The classical model of population growth suggests that the relative percentage increase in population is proportional with the time interval, i.e.

$$\frac{\Delta P(t)}{P(t)} = r\Delta t,$$

where the constant $r > 0$ denotes the population growth. Allowing for infinitesimal time intervals, the aforementioned equation writes as

$$dP(t) = rP(t)dt.$$

This differential equation has the solution $P(t) = P_0 e^{rt}$, where P_0 is the initial population size. The evolution of the population is driven by its growth rate

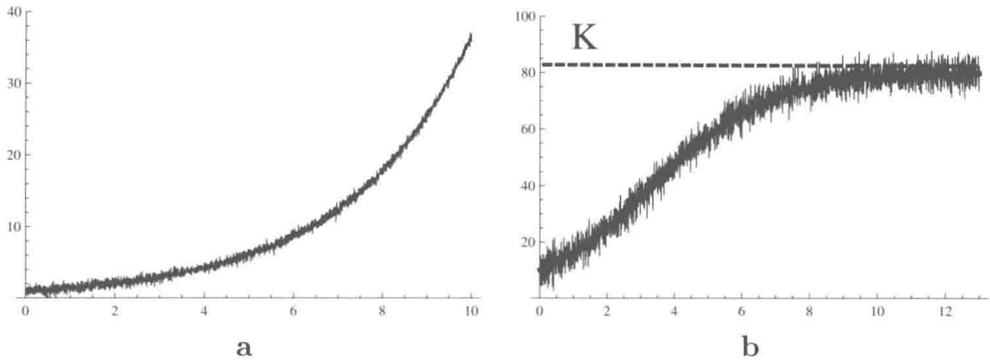


Figure 1.1: (a) *Noisy population with exponential growth.* (b) *Noisy population with logistic growth.*

r. In real life this rate is not constant. It might be a function of time t, or even more general, it might oscillate irregularly around some deterministic average function $a(t)$:

$$r_t = a(t) + \text{“noise”}.$$

In this case, r_t becomes a random variable indexed over time t. The associated equation becomes a stochastic differential equation

$$dP(t) = \big(a(t) + \text{“noise”}\big)P(t)dt. \tag{1.1.1}$$

Solving an equation of type (1.1.1) is a problem of Stochastic Calculus, see Fig. 1.1(a).

Logistic growth model The previous exponential growth model allows the population to increase indefinitely. However, due to competition, limited space and resources, the population will increase slower and slower. This model was introduced by P.F. Verhust in 1832 and rediscovered by R. Pearl in the twentieth century. The main assumption of the model is that the amount of competition is proportional with the number of encounters between the population members, which is proportional with the square of the population size

$$dP(t) = rP(t)dt - kP(t)^2dt. \tag{1.1.2}$$

The solution is given by the logistic function

$$P(t) = \frac{P_0K}{P_0 + (K - P_0)e^{-rt}},$$

where $K = r/k$ is the saturation level of the population. One of the stochastic variants of equation (1.1.2) is given by

$$dP(t) = rP(t)dt - kP(t)^2dt + \beta(\text{“noise”})P(t),$$