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Classical Recursion Theory

Volume II

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CLASSICAL RECURSION THEORY

VOLUME II

To my parents,
on their 50th anniversary.

Preface to Volume II

If it is bad luck to title a book "Volume I", as Gian-Carlo Rota says in *Indiscreet Thoughts*, it is not good luck to promise a Volume II in the introduction, either. For the ten years after the appearance of *Classical Recursion Theory* in 1989, I was torn between the easy choice of publishing a new edition of the first volume with an amended introduction, and the much harder choice of completing the second volume. Now that an apparently diverging sequence of successive drafts has finally come to a limit, I can release both a new edition of the first volume and a second volume.

In this last moment dedicated to thanksgivings, my first thoughts go to Laura. She entered my shattered life in 1986, putting its pieces back together and providing an inexhaustible flow of joy, happiness and understanding. She whipped me back into line, whenever I strayed from what she rightly considered the correct path. As I already wrote on her copy of Volume I, this book is also hers.

As the previous one, much of my last decade has been enlightened by visits to different parts of the world, made possible by a number of friends. First and foremost, Anil Nerode and Richard Shore, thanks to whom Ithaca became my second home for three years and thirteen summers. Then John Crossley in Melbourne in 1988, Dongping Yang in Beijing in 1992 and 1995, Gerald Sacks in Boston in 1995, 1996 and 1998, Antonin Kučera in Prague and Cristian Calude in Auckland in 1996, and Ding Decheng in Nanjing in 1998. Last but not least, Andrea Sorbi, whose selfless work as coordinator of a Human Capital and Mobility Project provided the funds for a number of European trips.

Even more than for the first volume, I owe a great debt to the colleagues who have read parts of the manuscript and have provided corrections and suggestions: Klaus Ambos-Spies, Francesco Bergadano, Cristian Calude, Barry Cooper, Ugo de' Liguoro, Lavinia Egidi, Dick Epstein, Matt Giorgi, Lane Hemachandra, Carl Jockusch, Martin Kummer, Antonin Kučera, Steffen Lempp, Bob Lubarsky, Wolfgang Merkle, Franco Montagna, Michael Morley, Dan Osherson, Alan Selman, Mark Simpson, Ted Slaman, Bob Soare, Andrea Sorbi, Frank Stephan, Helmut Veith and Yue Yang.

Whatever vision informs the book has been inspired by a handful of people, whose thoughts have provided and sustained inspiration through the years: Barry Cooper, Juris Hartmanis, Carl Jockusch, Georg Kreisel, Anil Nerode, Richard Platek and Gerald Sacks. Their acquaintance has been an undeserved honor, their teaching a much appreciated gift.

However, the book would never have been completed without the massive help of the Magnificent Four whose expectations have set my standards: Rod Downey, Bill Gasarch, André Nies and Richard Shore. As friends, they have provided a constant stimulus and encouragement, much needed in the face of hurdles and doubts. As colleagues, they have dedicated an enormous amount of time and energy to help me with explanations and proofs. If this volume does not displease them, I will be delighted.

As Beaumarchais once noticed, books are for authors as babies are for mothers: conceived in pleasure, carried with fatigue and given birth in pain. No words could better describe an enterprise that literally took away half of my life, nor better introduce the subject of dedication. Because, if I look back at my forty-eight years, in them I see only my parents more constant than this book. As a first child, I shared most of their life together: perhaps not as close and near as they would have liked, perhaps closer and nearer than they might have guessed. The deadline of their fiftieth anniversary on September 17th, 1999 has provided a major drive towards the completion of the book: another item to add to a long list of valuable parental offerings, which neither spoken nor printed words are able to match.

Ithaca - Torino
1989 - 1999

Contents

Introduction to Volume II

We obviously keep in Volume II the same notation and conventions used in Volume I. For the reader's convenience, we reproduce here the parts of the Introduction to Volume I which are relevant to Volume II.

What is in the Book

Recall that **Classical Recursion Theory is the study of real numbers or, equivalently, functions over the natural numbers**. The basic methods of analysis of the real numbers used in Volumes I and II are:

Hierarchies. A hierarchy is a stratification of a class of reals built from below, starting from a subclass that is taken as primitive (either because well understood, or because already previously analyzed), and obtained by iteration of an operation of class construction.

Degrees. Degrees are equivalence classes of reals under given equivalence relations, that identify reals with similar properties. Once a class of reals has been studied and understood, degrees are usually defined by identifying reals that look the same from that class point of view.

As might be imagined the two methods are complementary: first a class is analyzed in terms of intrinsic properties, for example by appropriately stratifying it in hierarchies, and then the whole structure of real numbers is studied modulo that analysis with the appropriate notion of degrees induced by the given class.

The previous complementarity is exploited throughout Volume II. In Chapter VIII we provide a study of **polynomial time computable functions** and of the induced notion of **polynomial time degrees** (similar notions of degrees could be introduced for most complexity classes studied in Chapter VIII). In Chapters IX, X and XIV we provide a study of the **recursively enumerable sets** and of the induced notion of **enumeration degrees**, while in Chapters

XII and XIII we provide a study of the **arithmetical sets** and of the induced notion of **arithmetical degrees**.

We now outline the skeleton of Volume II in more detail, referring to the introductions of the various chapters for more details. Chapters VII and VIII resume the analysis of the fundamental objects in Recursion Theory, the recursive sets and functions, and provide a microscopic picture of them. We start in Chapter VII with an abstract study of the complexity of computation of recursive functions. Then in Chapter VIII we attempt to build from below the world of recursive sets and functions that was previously introduced in just one go. A number of subclasses of interest from a computational point of view are introduced and discussed, among them: the **polynomial time (or space) computable functions** which provide an upper bound for the class of feasibly computable functions (as opposed to the abstractly computable ones); the **elementary functions**, which are the smallest known class of functions closed under time (deterministic or not) and space computations; the **primitive recursive functions**, which are those computable by the 'for' instruction of programming languages like PASCAL, i.e. with a preassigned number of iterations (as opposed to the recursive functions, computable by the 'while' instruction, which permits an unlimited number of iterations); the **ϵ_0-recursive functions**, which are those provably total in Peano Arithmetic.

Chapters IX and X return to the treatment of recursively enumerable sets. A good deal of information on their structure was already gathered in Chapter III, but here a systematic study of the structures of both the lattice of **recursively enumerable sets** and of the partial ordering of **recursively enumerable degrees** is undertaken. Special tools for their treatment are introduced, most prominent among them being the **priority method**, a constructive variation of the Baire Category method.

Chapter XI deals with **limit sets**, also known as Δ_2^0 sets, which are limits of recursive functions. They are a natural formalization of the notion of sets for which membership can be determined by effective trials and errors, unlike recursive sets (for which membership can be effectively determined), and recursively enumerable sets (for which membership can be determined with at most one mistake, by first guessing that an element is not in the set, and then changing opinion if it shows up during the generation of the set).

Chapter XII deals with **arithmetical sets**, which are definable in the language of First-Order Arithmetic. As a special tool for their treatment we introduce the method of **arithmetical forcing**, which can be combined with the Baire Category and the priority methods. Chapter XIII studies the structure of the continuum w.r.t. the notion of relative arithmetical definability provided by the **arithmetical degrees**, along the lines of Chapter V. Similarly, Chapter XIV studies the structure of the continuum w.r.t. a notion of relative recursive enumerability provided by the **enumeration degrees**.