GROUP THEORY IN PHYSICS

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Group Theory in Physics

To
Beatrice, Bruce, and Lei

PREFACE

Group theory provides the natural mathematical language to formulate symmetry principles and to derive their consequences in Mathematics and in Physics. The "special functions" of mathematical physics, which pervade mathematical analysis, classical physics, and quantum mechanics, invariably originate from underlying symmetries of the problem although the traditional presentation of such topics may not expressly emphasize this universal feature. Modern developments in all branches of physics are putting more and more emphasis on the role of symmetries of the underlying physical systems. Thus the use of group theory has become increasingly important in recent years. However, the incorporation of group theory into the undergraduate or graduate physics curriculum of most universities has not kept up with this development. At best, this subject is offered as a special topic course, catering to a restricted class of students! Symptomatic of this unfortunate gap is the lack of suitable textbooks on general group-theoretical methods in physics for all serious students of experimental and theoretical physics at the beginning graduate and advanced undergraduate level. This book is written to meet precisely this need.

There already exist, of course, many books on group theory and its applications in physics. Foremost among these are the old classics by Weyl, Wigner, and Van der Waerden. For applications to atomic and molecular physics, and to crystal lattices in solid state and chemical physics, there are many elementary textbooks emphasizing point groups, space groups, and the rotation group. Reflecting the important role played by group theory in modern elementary particle theory, many current books expound on the theory of Lie groups and Lie algebras with emphasis suitable for high energy theoretical physics. Finally, there are several useful general texts on group theory featuring comprehensiveness and mathematical rigor written for the more mathematically oriented audience. Experience indicates, however, that for most students, it is difficult to find a suitable modern introductory text which is both general and readily understandable.

This book originated from lecture notes of a general course on Mathematical Physics taught to all first-year physics graduate students at the University of Chicago and the Illinois Institute of Technology. The author is not, by any stretch of the imagination, an expert on group theory. The inevitable lack of authority and comprehensiveness is hopefully compensated by some degree of freshness in pedagogy which emphasizes underlying principles and techniques in ways easily appreciated by students. A number of ideas key to the power and beauty of the group theoretical approach are highlighted throughout the book, e.g., invariants and invariant operations; projection operators on function-, vector-, and operator-spaces; orthonormality and completeness properties of representation functions,..., etc. These fundamental features are usually not discussed or emphasized in the more practical elementary texts. Most books written by experts, on the other hand, either are "over the head" of the average student; or take many conceptual points for granted, thus leaving students to their own devices. I make a special effort to elucidate the important group theoretical methods by referring frequently to analogies in elementary topics and techniques familiar to students from basic courses of mathematics and physics. On the rich subject of Lie groups, key ideas are first introduced in the context of simpler groups using easily understandable examples. Only then are they discussed or developed for the more general and more complex cases. This is, of course, in direct contrast to the deductive approach, proceeding from the most abstract (general) to the more concrete (specific), commonly found in mathematical texts. I believe that the motivation provided by concrete examples is very important in developing a real understanding of the abstract theory. The combination of inductive and deductive reasoning adopted in our presentation should be closer to the learning experience of a student (as well as to the process of generalization involved in the creation of the theory by the pioneers) than a purely deductive one.

This book is written primarily for physicists. In addition to stressing the physical motivations for the formalism developed, the notation adopted is close to that of standard physics texts. The main subject is, however, the mathematics of group representation theory, with all its inherent simplicity and elegance. Physical arguments, based on well-known classical and quantum principles, are used to motivate the choice of the mathematical subjects, but not to interfere with their logical development. Unlike many other books, I refrain from extensive coverage of applications to specific fields in physics. Such diversions are often distracting for the coherent presentation of the mathematical theory; and they rarely do justice to the specific topics treated. The examples on physical applications that I do use to illustrate advanced group-theoretical techniques are all of a general nature applicable to a wide range of fields such as atomic, nuclear, and particle physics. They include the classification of arbitrary quantum mechanical states and general scattering amplitudes involving particles with spin (the Jacob-Wick helicity formalism), multipole moments and radiation for electromagnetic transitions in any physical system,..., etc. In spite of their clear group-theoretical origin and great practical usefulness, these topics are rarely discussed in texts on group theory.

Group representation theory is formulated on linear vector spaces. I assume the reader to be familiar with the theory of linear vector spaces at the level required for a standard course on quantum mechanics, or that of the classic book by Halmos. Because of the fundamental importance of this background subject, however, and in order to establish an unambiguous set of notations, I provide a brief summary of notations in Appendix I and a systematic review of the theory of finite dimensional vector spaces in Appendix II. Except for the most well-prepared reader, I recommend that the material of these Appendices be carefully scanned prior to the serious studying of the book proper. In the main text, I choose to emphasize clear presentation of underlying ideas rather than strict mathematical rigor. In particular, technical details that are needed to complete specific proofs, but are otherwise of no general implications, are organized separately into appropriate Appendices.

The introductory Chapter encapsulates the salient features of the group-theoretical approach in a simple, but non-trivial, example—discrete translational symmetry on a one dimensional lattice. Its purpose is to illustrate the flavor and the essence of this approach before the reader is burdened with the formal development of the full formalism. Chapter 2 provides an introduction to basic group theory. Chapter 3 contains the standard group representation theory. Chapter 4 highlights general properties of irreducible sets of vectors and operators which are used throughout the book. It also introduces the powerful projection operator techniques and the Wigner-Eckart Theorem (for any group), both of which figure prominently in all applications. Chapter 5 describes the representation theory of the symmetric (or permutation) groups with the help of Young tableaux and the associated Young symmetrizers. An introduction to symmetry classes of tensors is given, as an example of useful applications of the symmetric group and as preparation for the general representation theory of classical linear groups to be discussed later. Chapter 6 introduces the basic elements of representation theory of continuous groups in the Lie algebra approach by studying the one-parameter rotation and translation groups. Chapter 7 contains a careful treatment of the rotation group in three-dimensional space, SO(3). Chapter 8 establishes the relation between the groups SO(3) and SU(2), then explores several important advanced topics: invariant integration measure, orthonormality and completeness of the *D*-functions, projection operators and their physical applications, differential equations satisfied by the *D*-functions, relation to classical special functions of mathematical physics, group-theoretical interpretation of the spherical harmonics, and multipole radiation of the electromagnetic field. These topics are selected to illustrate the power and the breadth of the group-theoretical approach, not only for the special case of the rotation group, but as the prototype of similar applications for other Lie groups. Chapter 9 explores basic techniques in the representation theory of inhomogeneous groups. In the context of the simplest case, the group of motions (Euclidean group) in two dimensions, three different approaches to the problem are introduced: the Lie algebra, the induced representation, and the group contraction methods. Relation of the group representation functions to Bessel functions is established and used to elucidate properties of the latter. Similar topics for the Euclidean group in three dimensions are then discussed. Chapter 10 offers a systematic derivation of the finite-dimensional and the unitary representations of the Lorentz group, and the unitary representations of the Poincaré group. The latter embodies the full continuous space-time symmetry of Einstein's special relativity which underlies contemporary physics (with the exception of the theory of gravity). The relation between finite-dimensional (non-unitary) representations of the Lorentz group and the (infinite-dimensional) unitary representations of the Poincaré group is discussed in detail in the context of relativistic wave functions (fields) and wave equations. Chapter 11 explores space inversion symmetry in two, and three-dimensional Euclidean space, as well as four-dimensional Minkowski space. Applications to general scattering amplitudes and multipole radiation processes are considered. Chapter 12 examines in great detail new issues raised by time reversal invariance and explores their physical consequences. Chapter 13 builds on experience with the various groups studied in previous chapters and develops the general tensorial method for deriving all finite dimensional representations of the classical linear groups GL(m; C), GL(m; R), U(m, n), SL(m; C), SU(m, n), O(m, n; R), and SO(m, n; R). The important roles played by invariant tensors, in defining the groups and in determining the irreducible representations and their properties, is emphasized.

It may be noticed that, point and space groups of crystal lattices are conspicuously missing from the list of topics described above. There are two reasons for this omission: (i) These groups are well covered by many existing books emphasizing applications in solid state and chemical physics. Duplication hardly seems necessary; and (ii) The absence of these groups does not affect the coherent development of the important concepts and techniques needed for the main body of the book. Although a great deal of emphasis has been placed on aspects of the theory of group representation that reveal its crucial links to linear algebra, differential geometry, and harmonic analysis, this is done only by means of concrete examples (involving the rotational, Euclidean, Lorentz, and Poincare groups). I have refrained from treating the vast and rich general theory of Lie groups, as to do so would require a degree of abstraction and mathematical sophistication on the part of the reader beyond that expected of the intended audience. The material covered here should provide a solid foundation for those interested to pursue the general mathematical theory, as well as the burgeoning applications in contemporary theoretical physics, such as various gauge symmetries, the theory of gravity, supersymmetries, supergravity, and the superstring theory.

When used as a textbook, Chapters 1 through 8 (perhaps parts of Chapter 9 as well) fit into a one-semester course at the beginning graduate or advanced undergraduate level. The entire book, supplemented by materials on point groups and some general theory of Lie groups if desired, is suitable for use in a two-semester course on group theory in physics. This book is also designed to be used for self-study. The bibliography near the end of the book comprises commonly available books on group theory and related topics in mathematics and physics which can be of value for reference and for further reading.

My interest in the theory and application of group representations was developed during graduate student years under the influence of Loyal Durand, Charles Sommerfield, and Feza Gürsey. My appreciation of the subject has especially been inspired by the seminal works of Wigner, as is clearly reflected in the selection of topics and in their presentation. The treatment of finite-dimensional representations of the classical groups in the last chapter benefited a lot from a set of informal but incisive lecture notes by Robert Geroch.

It is impossible to overstate my appreciation of the help I have received from many sources which, together, made this book possible. My colleague and friend Porter Johnson has been extremely kind in adopting the first draft of the manuscript for field-testing in his course on mathematical physics. I thank him for making many suggestions on improving the manuscript, and in combing through the text to uncover minor grammatical flaws that still haunt my writing (not being blessed with a native English tongue). Henry Frisch made many cogent comments and suggestions which led to substantial improvements in the presentation of the crucial initial chapters. Debra Karatas went through the entire length of the book and made invaluable suggestions from a student's point of view. Si-jin Qian provided valuable help with proof-reading. And my son Bruce undertook the arduous task of typing the initial draft of the whole book during his busy and critical senior year of high school, as well as many full days of precious vacation time from college. During the period of writing this book, I have been supported by the Illinois Institute of Technology, the National Science Foundation, and the Fermi National Accelerator Laboratory.

Finally, with the deepest affection, I thank all members of my family for their encouragement, understanding, and tolerance throughout this project. To them, I dedicate this book.

Chicago
December, 1984

WKT

CONTENTS

CHAPTER 1

INTRODUCTION

Symmetry, Quantum Mechanics, Group Theory, and Special Functions in a Nutshell

The theory of group representation provides the natural mathematical language for describing symmetries of the physical world. Although the mathematics of group theory and the physics of symmetries were not developed simultaneously—as in the case of calculus and mechanics by Newton—the intimate relationship between the two was fully realized and clearly formulated by Wigner and Weyl, among others, before 1930. This close connection is most apparent in the framework of the new quantum mechanics. But much of classical physics, involving symmetries of one kind or another, can also be greatly elucidated by the group-theoretical approach. Specifically, the solutions to equations of classical mathematical physics and "state vectors" of quantum mechanical systems both form linear vector spaces. Symmetries of the underlying physical system require distinctive regularity structures in these vector spaces. These distinctive patterns are determined purely by the group theory of the symmetry and are independent of other details of the system.

Therefore, in addition to furnishing a powerful tool for studying new mathematical and physical problems, the group theoretical approach also adds much insight to the wealth of old results on classical "special functions" of mathematical physics previously derived from rather elaborate analytic methods. Since the 1950's, the application of group theory to physics has become increasingly important. It now permeates every branch of physics, as well as many areas of other physical and life sciences. It has gained equal importance in exploring "internal symmetries" of nature (such as isotopic spin and its many generalizations) as in elucidating traditional discrete and continuous space-time symmetries.

In this introductory chapter we shall use a simple example to illustrate the close relationship between physical symmetries, group theory, and special functions. This is done before entering the formal development of the next few chapters, so that the reader will be aware of the general underlying ideas and the universal features of the group theoretical approach, and will be able to see through the technical details which lie ahead. As with any "simple example", the best one can do is to illustrate the basic ideas in their most transparent setting. The full richness of the subject and the real power of the approach can be revealed only after a full exposition of the theory and its applications.

Since we shall try to illustrate the full scope of concepts with this example, notions of classical and quantum physics as well as linear vector spaces and Fourier analysis are all involved in the following discussion. For readers approaching this subject for the first time, a full appreciation of all the ideas may be more naturally attained by