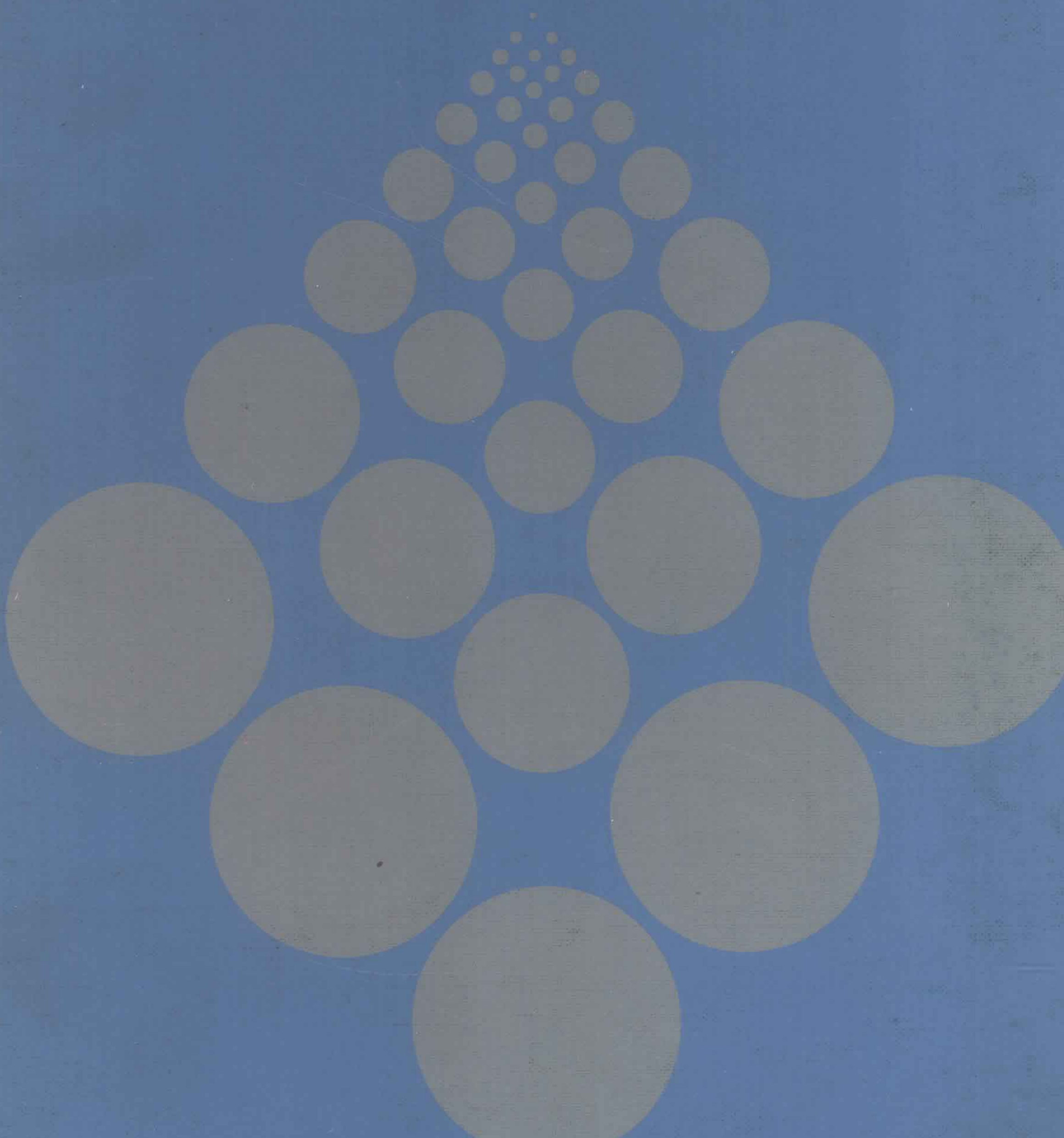
Calculus

Arthur B. Simon



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ARTHUR B. SIMON
Northwestern University

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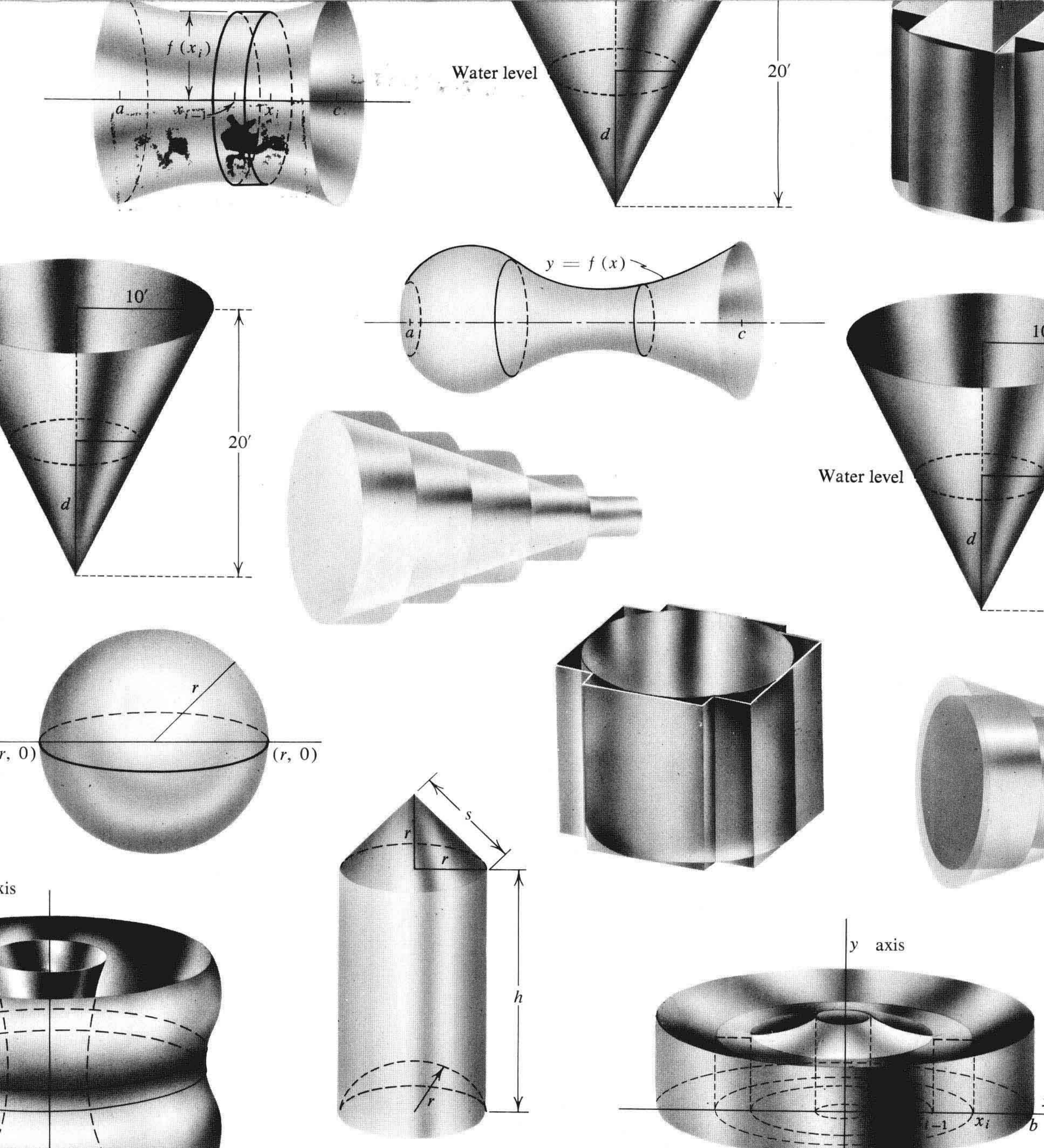

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PREFACE

The material in this book is intended as a basis for a four-quarter or three-semester beginning calculus course. It is designed to take advantage of the modern methods employed in teaching mathematics in high school.

Although each important concept is introduced intuitively, the approach, in general, is one of rigor. I believe that each student, whether a math major or not, *deserves the opportunity to be challenged*—not only by applications in the form of exercises, but also by the concepts themselves. The only hope, then, is a rigorous treatment.

That is not to say that every reader could or should be expected to comprehend fully every idea presented here. Indeed, I am convinced that for complete understanding, the student needs several exposures to the basic concepts of calculus, and the book is written with this in mind. Thus the lecturer may, at his discretion, include or omit as much of the rigorous treatment as he deems necessary, without seriously disrupting the continuity of the presentation. The important thing is that the ideas are available for all who are interested.

The obvious disadvantage of a rigorous approach is that presenting the background material vital to a meaningful exposition necessitates a long delay in the introduction of the two main tools of calculus: differentiation and integration. I have tried to overcome this handicap with Chapter 2, entitled "Computational Calculus." Other appropriate titles would be "Axiomatic Calculus," "Algebraic Calculus," or even "Cookbook Calculus," for it is all of these. The elementary rules for differentiation and integration are listed as algebraic formulas. The sections that follow help the student to acquire a dexterity in handling these formulas and to use them in computing such things as maximum and minimum, velocity and acceleration, straight-line motion, areas, and work. The student is cautioned throughout the chapter that he is merely learning some of the *uses* of calculus and that in later chapters he will learn *why* the methods work. Some of the advantages of this approach are that it provides (1) a short (about three weeks) over-all view of applications of differentiation *and* integration, (2) an early opportunity to develop manipulative skills that permit the student to concentrate on the theory at the proper time, (3) a convenient break in the theoretical nature of Chapters 1 and 3, (4) an opportunity for the material of Chapter 1 to solidify with the help of recall by a system of "Review Exercises," (5) a good review for the student who has "had calculus" in high school and an introduction to the subject for the other students, and (6) the proper setting for simple, straightforward exercises; more complex problems are presented in the chapters dealing with the theory.

The approach to multidimensional calculus is a modern one. Chapter 11 is a short introduction to linear algebra which covers the concepts used in later chapters; (a) linear transformations and their relationship to matrices, and (b) determinants and their relationship to area and volume. In Chapter 12 the differential is defined as a linear transformation. Chapter 13 presents multiple and line integrals.

Very little else need be said about the material of the text, but perhaps a few words about the ordering of the chapters would be appropriate. I suggest taking each chapter in turn, with the possible exception of Chapter 10, which treats vectors. Actually, it could come anywhere after Chapter 6. In teaching the course myself, I used Chapter 10 simultaneously as a sort of review of calculus and as an introduction to multidimensional calculus, and so it appears where it does. Although I strongly advise against it, Chapter 2 may be omitted. However, since the easy, manipulative type of exercises are for the most part confined to Chapter 2, its omission presents a mild problem. This difficulty is surmounted by directing the reader to the appropriate exercises in Chapter 2 which pertain to the theory just discussed.

Exercises form an important part of any test and this book is no exception. The exercises at the end of each section are graded; the higher-numbered ones are usually more difficult. The problems marked with asterisks (*) are probably the most difficult. *A black dot (•) means the indicated problem is important and hence deserves special attention.*

There is a device used in connection with the exercises which, to my knowledge, is an innovation: the Review Exercises. In most sections, after the regular set of problems, there is a list of Review Exercises which concerns previously covered material. In this way, the reader is constantly reviewing. These exercises are also used as a convenient method of recalling a concept studied earlier and needed in the next lesson. One disadvantage of this system is that the exercises concerning one topic do not all appear in one place. In cases where this situation is detrimental, the Review Exercises pertaining to a certain section appear as exercises in that section with references to where they are also listed as Review Exercises. For example, after Section 4.7 there is Exercise 20 (same as 5.6.R4). Thus the student may work Exercise 20 in the set after Section 4.7, or work it as Review Exercise R4 after Section 5.6. In teaching from this book, I made the Review Exercises an integral part of the course and found them to be quite effective.

I would like to thank the many people who have been involved in the writing of this book. Foremost among them are the students who used the manuscript as a text and whose comments were extremely helpful to me. I am also indebted to Professors Carl Allendoerfer and Philip Ostrand for reading the preliminary versions and making valuable suggestions, and to Vera Fisher for typing the manuscript. Finally, I should like to express deep gratitude to my entire family for their patience and understanding, and in particular to my perspicacious wife, who made me promise never to write another calculus book.

A. B. S.

COMMENTS TO THE INSTRUCTOR ON CHAPTER 2

Teaching Chapter 2 in the order in which it appears actually works! This conclusion is drawn not only from my own experience but that of my colleagues here and, as evidenced by letters I have received, that of mathematicians at other institutions who are currently using the earlier version entitled *First-Year Calculus.*

In the preface I listed six advantages of including Chapter 2. A seventh and, perhaps, most important advantage is that it develops *self-confidence* in the student. Almost immediately he is actually working calculus problems and applying them to physical situations! Your exam covering this material should have a very high median. This means the student can attack the theory with the assurance of being able to work problems and also with a reservoir of good grades.

I suppose the most effective method of teaching Chapter 2 is to draw the usual diagrams which indicate why the derivative can be interpreted as "rate of change" or "slope" (see, for example, Figure 1 of Section 4.2) and then use them to justify the applications to velocity-acceleration and maximum-minimum problems. The same technique can be used with integration. These diagrams and explanations are not included in the text because I did not want to give them "official" status before discussing limits.†

† Personally, I prefer to treat the formulas and rules in Chapter 2 as though they were the accumulation of generations of mystical thought and present them without any justification. I used this method last year and *the students actually came begging to learn the theory* (a phenomenon I had never experienced before). I assured them that soon they would understand everything, and when "enlightenment" finally came I think they were genuinely excited. This is the kind of thing I think we should all be striving to introduce into our math courses—more interest, more excitement, more fun.

Finally, I should point out that, after all, Chapter 2 can be omitted in the presentation of the course and used only for its exercises. The book is flexible enough to do this but I would recommend it only if the students are known to have an adequate background in analytic geometry, trigonometry, *and* computational calculus. Furthermore, the review exercises cited on page 48 which cover an informal introduction to continuity should be assigned and discussed in any case.

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CHAPTER 14 DIFFERENTIAL EQUATIONS

CHAPTER I

SETS AND FUNCTIONS

1.1. SETS

In the study of geometry the concepts of point, line, and angle are left undefined. Then by a system of axioms, definitions, and theorems, we are able to develop a set of rules that govern the behavior of points and lines and, more important, allow us to create new objects and concepts from the old, familiar ones. The same procedure will be adopted here.

The concepts of *element* and *set* are left undefined. These words will be used so often that we need some synonyms for literary purposes; *element*, *member*, and *point* have identical connotation, and *set*, *collection*, and *family* all mean the same thing.

The chairs in a given classroom, a bouquet of flowers, the even integers, and a flock of sheep are all examples of sets. A particular chair, one of the flowers, the number 4, and the black one are elements of the given sets.

A set as a mathematical object is useless unless there is a systematic method, practical or not, for determining its members. Thus we rule out "the family consisting of the three greatest living composers," because it would be unlikely that we could agree upon its members. On the other hand, we admit "the family consisting of the living composers whose work has been performed by the Chicago Symphony Orchestra."

We shall occasionally want to consider sets whose elements are themselves sets—that is, sets of sets, sets of sets of sets, and so on. For example, the United Nations is a set whose points are member nations. Some of these nations are themselves collections whose elements are states; some states are a family of counties; and finally, a county is a set of citizens.

If A is a set and x is a member of A, we indicate this by writing

$$x \in A,$$

which is read "x is an element of A."

If A is a set and x is an object that is not a member of A, we write

$$x \notin A,$$

which is read "x is not an element of A."

Let N be the set "United Nations," U the set "United States," I the set "Illinois," and C the set "Cook County," and let M be a single citizen living in Cook County. Then, of course, $M \in C$, $C \in I$, $I \in U$, and $U \in N$. It is important to note that $M \notin I$, even though $M \in C$ and $C \in I$, because I is a state and states were defined above as sets of counties; clearly, M is not a county. For similar reasons $I \notin N$, even though $I \in U$ and $U \in N$.

There are two ways of specifying a set. One method is by actually *listing* all the names or symbols that represent the members and the other is by *describing* the elements in words or symbols. A listing of the set U above would be an enumeration of the fifty states and would be written

$$U = \{\text{Maine, Vermont, New Hampshire}, \ldots\}.$$

We do not have the room or the inclination to write them all. A description can be written

$$U = \{\text{all states in the United States}\}.$$

As another example, let A be the set of all even integers bigger than 1 but smaller than 9. A listing of A would be

$$A = \{2, 4, 6, 8\},$$

which is read "A is the set consisting of the elements 2, 4, 6, and 8." The order in which they appear is unimportant, as we shall see below. To give a description of A, we introduce the notation that is standard:

$$A = \{x : x \text{ is an even integer and } 1 < x < 9\}.$$

This notation is read "A is the set of all x such that x is an even integer and 1 is less than x and x is less than 9." You can see why we use symbolic notation rather than the English sentence. The symbol "$\{x$" is read "the set of all x"; the colon ":" is read "such that"; and the sentence that follows is the description or *defining property*. Thus *an object is an element of A if and only if it is in a listing of A or satisfies the defining property of A.*

Suppose $A = \{x : x \text{ is a weekday that begins with } T\}$ and $B = \{y : y \text{ is the weekday that precedes Wednesday or precedes Friday}\}$. Are A and B the same set even though they have different defining properties? If we say two sets are equal if and only if their defining properties are (word for word) identical, then A and B are not the same. To avoid the obvious drawbacks that would result, mathematicians have settled on the following definition.

1.1.1. Definition

Two sets X and Y are *equal* and written $X = Y$ if and only if they have the same (that is, identical) elements. If X is *not equal* to Y, we write $X \neq Y$.

Thus in the example above $A = B$, because their members are identical—namely, Tuesday and Thursday. Similarly, if $C = \{\text{Tuesday, Thursday}\}$, then $A = C$ and $B = C$.

The most important set in this course is the set of all real numbers. *The letter R will be used throughout to denote this set.* Some of the familiar properties of R are set forth below.

The *natural* (or *counting*) numbers are the elements of the set $N = \{1, 2, 3, \ldots\}$. (This notation stands for a listing of the set N; the numbers 1, 2, and 3 establish a pattern, and the "dots," or ellipsis, after 3 indicate the pattern is to be continued.) Each element of N is an element of R.

The elements of the set I consisting of the natural numbers together with their negatives and zero are called *integers*. In symbols, $I = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$. Every integer is a real number.

The *rational numbers* are the elements of the set F (F for "fraction") of all numbers which may be represented in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Examples are $\frac{1}{2}$, $\frac{-7}{3}$, and $\frac{13}{6}$. Every rational number is in R.

Notice, every natural number is an integer and every integer is a rational number; for example, $5 = \frac{5}{1}$ and $0 = \frac{0}{8}$.

We assume that you are well acquainted with the following concepts:

1. The geometric representation of R as a line called the (real) *number line* (Figure 1). Every point on the line represents a real number, and every real number can be located or represented as a point on the line.†

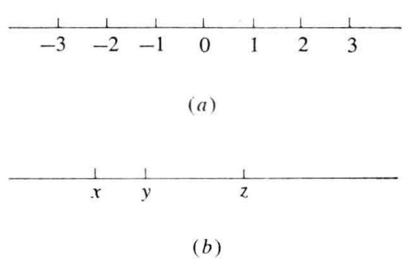

Figure 1. The number line. (a) The (real) number line. (b) $x < y$ and $y < z$.

† Although the number line and the set R are distinct objects, it is common practice to treat their respective elements as one and the same. Thus instead of writing "the point representing the real number 1," we shall often write "the point 1."