BIRKHÄUSER

CORNERSTONES

PARTIAL DIFFERENTIAL EQUATIONS

Second Edition

Emmanuele DiBenedetto

Emmanuele DiBenedetto

Partial Differential Equations

Second Edition

Birkhäuser
Boston • Basel • Berlin

Emmanuele DiBenedetto
Department of Mathematics
Vanderbilt University
Nashville, TN 37240
USA
em.diben@vanderbilt.edu

ISBN 978-0-8176-4551-9 e-ISBN 978-0-8176-4552-6
DOI 10.1007/978-0-8176-4552-6
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2009938184

Mathematics Subject Classification (2000): 31B05, 31B20, 35A10, 35B45, 35B65, 35D10, 35J05, 35K05, 35L05, 35L60, 35L65, 45A05, 45B05, 45C05, 49J40

Printed on acid-free paper

Birkhäuser Boston is part of Springer Science+Business Media (www.birkhauser.com)

BIRKHÄUSER

Cornerstones

Preface to the Second Edition

This is a revised and extended version of my 1995 elementary introduction to partial differential equations. The material is essentially the same except for three new chapters. The first (Chapter 8) is about non-linear equations of first order and in particular Hamilton–Jacobi equations. It builds on the continuing idea that PDEs, although a branch of mathematical analysis, are closely related to models of physical phenomena. Such underlying physics in turn provides ideas of solvability. The Hopf variational approach to the Cauchy problem for Hamilton–Jacobi equations is one of the clearest and most incisive examples of such an interplay. The method is a perfect blend of classical mechanics, through the role and properties of the Lagrangian and Hamiltonian, and calculus of variations. A delicate issue is that of identifying "uniqueness classes." An effort has been made to extract the geometrical conditions on the graph of solutions, such as quasi-concavity, for uniqueness to hold.

Chapter 9 is an introduction to weak formulations, Sobolev spaces, and direct variational methods for linear and quasi-linear elliptic equations. While terse, the material on Sobolev spaces is reasonably complete, at least for a PDE user. It includes all the basic embedding theorems, including their proofs, and the theory of traces. Weak formulations of the Dirichlet and Neumann problems build on this material. Related variational and Galerkin methods, as well as eigenvalue problems, are presented within their weak framework. The Neumann problem is not as frequently treated in the literature as the Dirichlet problem; an effort has been made to present the underlying theory as completely as possible. Some attention has been paid to the local behavior of these weak solutions, both for the Dirichlet and Neumann problems. While efficient in terms of existence theory, weak solutions provide limited information on their local behavior. The starting point is a sup bound for the solutions and weak forms of the maximum principle. A further step is their local Hölder continuity.

An introduction to these local methods is in Chapter 10 in the framework of DeGiorgi classes. While originating from quasi-linear elliptic equations, these classes have a life of their own. The investigation of the local and boundary behavior of functions in these classes, involves a combination of methods from PDEs, measure theory, and harmonic analysis. We start by tracing them back to quasi-linear elliptic equations, and then present in detail some of these methods. In particular, we establish that functions in these classes are locally bounded and locally Hölder continuous, and we give conditions for the regularity to extend up to the boundary. Finally, we prove that non-negative functions on the DeGiorgi classes satisfy the Harnack inequality. This, on the one hand, is a surprising fact, since these classes require only some sort of Caccioppoli-type energy bounds. On the other hand, this raises the question of understanding their structure, which to date is still not fully understood. While some facts about these classes are scattered in the literature, this is perhaps the first systematic presentation of DeGiorgi classes in their own right. Some of the material is as recent as last year. In this respect, these last two chapters provide a background on a spectrum of techniques in local behavior of solutions of elliptic PDEs, and build toward research topics of current active investigation.

The presentation is more terse and streamlined than in the first edition. Some elementary background material (Weierstrass Theorem, mollifiers, Ascoli–Arzelá Theorem, Jensen's inequality, etc..) has been removed.

I am indebted to many colleagues and students who, over the past fourteen years, have offered critical suggestions and pointed out misprints, imprecise statements, and points that were not clear on a first reading. Among these Giovanni Caruso, Xu Guoyi, Hanna Callender, David Petersen, Mike O'Leary, Changyong Zhong, Justin Fitzpatrick, Abey Lopez and Haichao Wang. Special thanks go to Matt Calef for reading carefully a large portion of the manuscript and providing suggestions and some simplifying arguments. The help of U. Gianazza has been greatly appreciated. He has read the entire manuscript with extreme care and dedication, picking up points that needed to be clarified. I am very much indebted to Ugo.

I would like to thank Avner Friedman, James Serrin, Constantine Dafermos, Bob Glassey, Giorgio Talenti, Luigi Ambrosio, Juan Manfredi, John Lewis, Vincenzo Vespri, and Gui Qiang Chen for examining the manuscript in detail and for providing valuable comments. Special thanks to David Kinderlehrer for his suggestion to include material on weak formulations and direct methods. Without his input and critical reading, the last two chapters probably would not have been written. Finally, I would like to thank Ann Kostant and the entire team at Birkhäuser for their patience in coping with my delays.

Vanderbilt University
June 2009

Emmanuele DiBenedetto

Preface to the First Edition

These notes are meant to be a self contained, elementary introduction to partial differential equations (PDEs). They assume only advanced differential calculus and some basic L^p theory. Although the basic equations treated in this book, given its scope, are linear, I have made an attempt to approach then from a non-linear perspective.

Chapter I is focused on the Cauchy–Kowalewski theorem. We discuss the notion of characteristic surfaces and use it to classify partial differential equations. The discussion grows from equations of second-order in two variables to equations of second-order in N variables to PDEs of any order in N variables.

In Chapters 2 and 3 we study the Laplace equation and connected elliptic theory. The existence of solutions for the Dirichlet problem is proven by the Perron method. This method clarifies the structure of the sub(super)-harmonic functions, and it is closely related to the modern notion of *viscosity solution*. The elliptic theory is complemented by the Harnack and Liouville theorems, the simplest version of Schauder's estimates, and basic L^p-potential estimates. Then, in Chapter 3 the Dirichlet and Neumann problems, as well as eigenvalue problems for the Laplacian, are cast in terms of integral equations. This requires some basic facts concerning double-layer potentials and the notion of compact subsets of L^p, which we present.

In Chapter 4 we present the Fredholm theory of integral equations and derive necessary and sufficient conditions for solving the Neumann problem. We solve eigenvalue problems for the Laplacian, generate orthonormal systems in L^2, and discuss questions of completeness of such systems in L^2. This provides a theoretical basis for the method of separation of variables.

Chapter 5 treats the heat equation and related parabolic theory. We introduce the representation formulas, and discuss various comparison principles. Some focus has been placed on the uniqueness of solutions to the Cauchy problem and their behavior as $|x| \to \infty$. We discuss Widder's theorem and the structure of the non-negative solutions. To prove the parabolic Harnack estimate we have used an idea introduced by Krylov and Safonov in the context of fully non-linear equations.

The wave equation is treated in Chapter 6 in its basic aspects. We derive representation formulas and discuss the role of the characteristics, propagation of signals, and questions of regularity. For general linear second-order hyperbolic equations in two variables, we introduce the Riemann function and prove its symmetry properties. The sections on Goursat problems represent a concrete application of integral equations of Volterra type.

Chapter 7 is an introduction to conservation laws. The main points of the theory are taken from the original papers of Hopf and Lax from the 1950s. Space is given to the minimization process and the meaning of taking the initial data in the sense of L^1. The uniqueness theorem we present is due to Kruzhkov (1970). We discuss the meaning of *viscosity solution* vis-à-vis the notion of sub-solutions and maximum principle for parabolic equations. The theory is complemented by an analysis of the asymptotic behavior, again following Hopf and Lax.

Even though the layout is theoretical, I have indicated some of the physical origins of PDEs. Reference is made to potential theory, similarity solutions for the porous medium equation, generalized Riemann problems, etc.

I have also attempted to convey the notion of *ill-posed* problems, mainly via some examples of Hadamard.

Most of the background material, arising along the presentation, has been stated and proved in the complements. Examples include the Ascoli–Arzelà theorem, Jensen's inequality, the characterization of compactness in L^p, mollifiers, basic facts on convex functions, and the Weierstrass theorem. A book of this kind is bound to leave out a number of topics, and this book is no exception. Perhaps the most noticeable omission here is some treatment of numerical methods.

These notes have grown out of courses in PDEs I taught over the years at Indiana University, Northwestern University and the University of Rome II, Italy. My thanks go to the numerous students who have pointed out misprints and imprecise statements. Of these, special thanks go to M. O'Leary, D. Diller, R. Czech, and A. Grillo. I am indebted to A. Devinatz for reading a large portion of the manuscript and for providing valuable critical comments. I have also benefited from the critical input of M. Herrero, V. Vesprii, and J. Manfredi, who have examined parts of the manuscript. I am grateful to E. Giusti for his help with some of the historical notes. The input of L. Chierchia has been crucial. He has read a large part of the manuscript and made critical remarks and suggestions. He has also worked out in detail a large number of the problems and supplied some of his own. In particular, he wrote the first draft of problems **2.7–2.13** of Chapter 5 and **6.10–6.11** of Chapter 6. Finally I like to thank M. Cangelli and H. Howard for their help with the graphics.

Contents