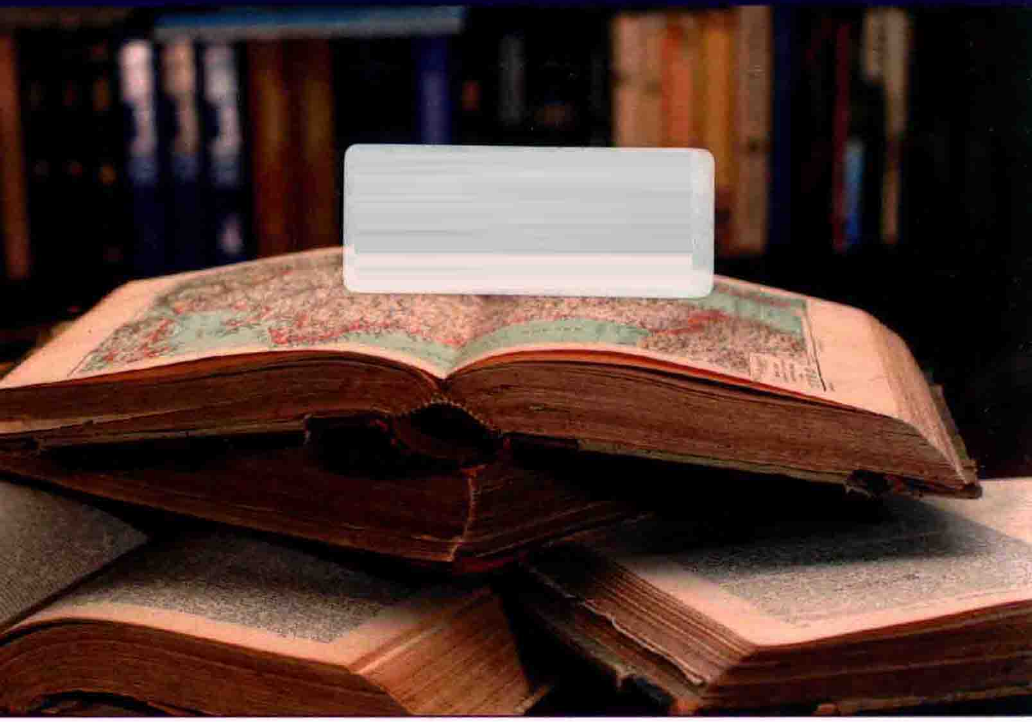

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CONTENTS

FUZZY AND FUZZY SOFT STRUCTURE IN ALGEBRA

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2012

CONTENTS

INTRODUCTION

Researchers in economics, engineering, environmental science, sociology, medical science, and many other fields deal daily with the complexities of modeling uncertain data. Classical methods are not always succesfull, because the uncertaintes appearing in these domains may be of various types. To exceed these uncertainties, some kinds of theories were given like theory of fuzzy sets, intuitionistic fuzzy sets, rough sets, i.e. which we can use as mathematical tools for dealing with uncertainties. As was mentioned in [31], these theories have their own difficulties. In 1999, Molodsov initated a novel concept of soft set theory, which is a completely new approach for modeling vagueness and uncertainty.

A soft set is a parameterized family of subsets of the universal set. We can say that soft sets are neigh borhood systems, and that they are a special case of context-dependent fuzzy sets. In soft set theory the problem os setting the membership function, among other related problems, simply does not arise. This makes the theory very convenient and easy to apply in practice. Soft set theory has potential applications in many different fields, including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory, and measurement theory. Also Maji et al. presented the definition of fuzzy soft set and Roy et al. presented some applications of this notion to decision making problems.

The theory of fuzzy sets, first deveoped by Zadeh [7] in 1965, is perhaps the most appropriate framework for deaing uncertainties. A fuzzy set is defined by its membership function, whose values are defined on the closed interval [0,1]. Some or all elements o a fuzzy set are thus assigned only partial membership. The idea of extending the concept of fuzzy sets to algebra dates back to the introduction in 1971 by Rosenfeld of fuzzy subgroups of a group [6]. Negoita and Ralescu [2] introduced the notion of fuzzy modules. Since then several researchers have studied fuzzy modules and then Lopez-Permouth and Malik [5] have given the category of fuzzy modules which is shown the totion *R-fzmod*. Ameri and Zahedi

[12] defined the concept of fuzzy exact sequence in the category of fuzzy modules and obtained some results about these notions. Same researchers [19] have previosly introduced the category of fuzzy (co) chain complexes and determined fuzzy homology functor in the category. It was proved that this functor is invariant to fuzzy homotopy given in [19]. Generally, a sequence of fuzzy homology modules is not exact. Zahedi and Ameri defined a tensor product of fuzzy modules and derivative functor of the tensor product in [11]. Bayramov and Gunduz [45] have formed exact sequence of fuzzy homology modules under some conditions in category of fuzzy modules and by using this they proved the universal coefficient theorem in the category of *R-fzmod.* Bayramov and Gunduz [44] did some research about inverse limit and it"s derivative functor in the category of fuzzy modules. Cunduz and Davvaz [46] have broadened the universal coefficient theorem into the category of intuitionistic fuzzy modules.

The algebraic structure in soft set was firstly given by Aktas and Cagman [36]. They have showed that the soft group structure is invariant under the operations of soft groups on soft sets by defining soft groups. Gunduz and Bayramov proved the isomorphism theorems for soft groups [50]. After then in [38,39,40] semirings, rings and some properties of these in soft sets were given. In [37] fuzzy soft group was defined and some properties of it"s were researched. The isomorphism theorems for fuzzy soft groups were proven [51]. Bayramov and Gunduz have given the concepts of fuzzy soft modules and intuitionistic fuzzy soft modules and then they have ressearched some properties of them. In the studies [48,49] the existence and properties of the limit of inverse and direct systems in categories of soft modules and fuzzy soft modules were investigated. For this, there are some studies [52,53] which are about homology modules of chain complexes in these categories.

This book contains two sections.

In first section some researches about fuzzy modules are presented.

In the second section studies about soft modules and fuzzy soft modules are presented.

I. CATEGORY of FUZZY MODULES

1.1. Fuzzy Modules

Let R be an ordinary ring and M be a left (or right) R-module.

Definition 1.1.1 (M,μ) is called a fuzzy left R-module, if there is a map

$$\mu : M \to [0,1]$$

satisfying the following conditions:

(1) $\mu(a+b) \geq \min\{\mu(a),\mu(b)\}$ $(\forall a,b \in M)$,

(2) $\mu(-a) = \mu(a)$ $(\forall a \in M)$,

(3) $\mu(0) = 1$,

(4) $\mu(ra) \geq \mu(a)$ $(\forall a \in M, r \in R)$.

Definition 1.1.2. Let (M,μ) and (N,ν) be arbitrary fuzzy left R-modules. A fuzzy R-map $\tilde{f} : (M,\mu) \to (N,\nu)$ should satisfy the following conditions:

(1) $f : M \to N$ is an R-map,

(2) $\nu(f(a)) \geq \mu(a)$ $(\forall a \in M)$.

Lemma 1.1.3. Let $Hom((M,\mu),(N,\nu))$ express the set of all fuzzy R-maps of (M,μ) into (N,ν). Then $Hom((M,\mu),(N,\nu))$ is additive group. Moreover, it can be made into a left R-module if R commutative.

Proof. First of all, since $\nu(0(a)) = \nu(0) = 1 \geq \mu(a)\ (\forall a \in M)$, there exist the fuzzy R-map

$$\tilde{0} : (M, \mu) \to (N, \nu).$$

For any $\tilde{f}, \tilde{g} \in Hom\big((M,\mu),(N,\nu)\big)$, because

$$\nu((f+g)(a)) = \nu(f(a)+g(a)) \geq \min\{\nu(f(a)), \nu(g(a))\} \geq \\ \min\{\mu(a), \mu(a)\} = \mu(a)\ (\forall a \in M),$$

we can dwfine

$$\tilde{f} + \tilde{g} = \widetilde{f+g} \in Hom\big((M,\mu),(N,\nu)\big).$$

The addition obviosly satisfies the commutative law and associative law.
Also define

$$-\tilde{f} = \widetilde{-f} \text{ for any } \tilde{f} \in Hom\big((M,\mu),(N,\nu)\big).$$

We have confidence in the definition, because

$$\nu((-f)(a)) = \nu(-f(a)) = \nu(f(a)) \geq \mu(a)\ (\forall a \in M).$$

Precisely, $\tilde{f} + \tilde{0} = \tilde{0} + \tilde{f} = \tilde{f}$ and $\tilde{f} + \widetilde{-f} = \widetilde{-f} + \tilde{f} = \tilde{0}$. We have obtained the additive group $Hom\big((M,\mu),(N,\nu)\big)$.

Furthermore, we define R-scalar multiplication as follows. For every $\tilde{f} \in Hom\big((M,\mu),(N,\nu)\big)$ and $r \in R$, define

$$(r\tilde{f})(a) = \tilde{f}(ra)\ (\forall a \in M).$$

As the map $a \mapsto f(ra)$ is the ordinary R-map of M into N and

$$\nu((r\tilde{f})(a)) = \nu(\tilde{f}(ra)) \geq \mu(ra) \geq \mu(a).$$

it follows that $r\tilde{f} \in Hom\big((M,\mu),(N,\nu)\big)$. Assume that R is commutative ring. It is clear

$$(r(\tilde{f}+\tilde{g}))(a) = (r(\widetilde{f+g}))(a) = \widetilde{f+g}(ra) = (\tilde{f}+\tilde{g})(ra) \\ = (r\tilde{f})(a) + (r\tilde{g})(a) = (r\tilde{f} + r\tilde{g})(a),$$

$$((r_1+r_2)\widetilde{f})(a)=\widetilde{f}((r_1+r_2)a)=\widetilde{f}(r_1a+r_2a)=\widetilde{f}(r_1a)+\widetilde{f}(r_2a)$$
$$=(r_1\widetilde{f})(a)+(r_2\widetilde{f})(a)=(r_1\widetilde{f}+r_2\widetilde{f})(a),$$
$$((r_1r_2)\widetilde{f})(a)=\widetilde{f}((r_1r_2)a)=\widetilde{f}(r_2(r_1a))=(r_2\widetilde{f})(r_1a)=(r_1(r_2\widetilde{f}))(a),$$
$$1\cdot\widetilde{f}(a)=\widetilde{f}(1\cdot a)=\widetilde{f}(a).$$

Therefore

$$r(\widetilde{f}+\widetilde{g})=r\widetilde{f}+r\widetilde{g},(r_1+r_2)\widetilde{f}=r_1\widetilde{f}+r_2\widetilde{f},(r_1r_2)\widetilde{f}=r_1(r_2\widetilde{f}),1\widetilde{f}=\widetilde{f}.$$

We have proved that $Hom((M,\mu),(N,\nu))$ is a left R-module if R is commutative ring.

Next, given

$$\widetilde{f}:(M,\mu)\rightarrow(N,\nu) \text{ and } \widetilde{g}:(N,\nu)\rightarrow(S,\eta)$$

Because

$$\eta((gf)(a)=\eta(g(f(a))\geq\nu(f(a))\geq\mu(a)\ (\forall a\in M),$$

We can define the composition as follows:

$$\widetilde{g}\cdot\widetilde{f}=\widetilde{g\cdot f}.$$

Fuzzy modules and morphisms of their is consists of a category. This category is denoted $Fz-Mod$.

Lemma 1.1.4. Let M and N be $R-$modules and $f:M\longrightarrow N$ be an $R-$ homomorphism.

(i) If (M,μ) is a fuzzy module, there exist a modular grade fonksiyon μ^f on N such that for any modular grade fonksiyon η on N, $\widetilde{f}:(M,\mu)\rightarrow(N,\eta)$ is fuzzy homomorfism of fuzzy modules if and only if $\eta\geq\mu^f$.

(ii) If (N,η) is fuzzy $R-$module, there exist a modular grade function η_f on M such that for any fuzzy R-module (M,μ), $\widetilde{f}:(M,\mu)\longrightarrow(N,\eta)$ is fuzzy homomorfism of fuzzy modules if and only if $\mu\leq\eta_f$.

Here $\mu^f(y) = \sup\{\mu(x) : f(x) = y\}$, $\eta_f(x) = \eta(f(x))$.

Lemma 1.1.5. (i) Given modules $\{M_i : i \in I\}$ and N and a family of R-homomorphisms $A = \{f_i : M_i \longrightarrow N \mid i \in I\}$. If $\{(M_i, \mu_i) | i \in I\}$ are fuzzy modules, then there exist a smallest grade function $\eta = \mu^A = \mu^{\{f_i\}}$ on N such that, for all $i \in I$, $\tilde{f}_i : (M_i, \mu_i) \longrightarrow (M, \eta)$ is fuzzy homomorphism of fuzzy modules.

(ii) Given modules M and $\{ N_i | i \in I \}$ and a family of R-homomorphisms $B = \{ g_i : M \longrightarrow N_i \mid i \in I \}$. If (N_i, η_i) are fuzzy modules, then there exist a largest grade function $\mu = \eta_B = \eta_{\{g_i\}}$ such that for all $i \in I$, $\tilde{g}_i : (M, \mu) \longrightarrow (M_i, \eta_i)$ is fuzzy homomorphism of fuzzy modules.

Here $\eta = \mu^A = \vee_{i \in I} \mu_i^{f_i}$, $\mu = \eta_B = \wedge_{i \in I} (\eta_i)_{g_i}$.

By using this lemma, we define the concept of submodule, quotient module, product and coproduct operation in the category of fuzzy modules.

a) If (M, μ) is an fuzzy module and $N \subset M$ is a submodule, then $(N, \mu|_N)$ is fuzzy submodule of (M, μ).

b) If (M, μ) is an fuzzy module and $p : M \to M/_\sim$ is the canonical projection, then $(M/_\sim, \mu_p)$ is quotient module of (M, μ).

Hence, for each fuzzy homomorphism of fuzzy modules $\tilde{f} : (M, \mu) \to (N, \nu)$, the fuzzy submodule $(\ker f, \mu|_{\ker f})$ and $(N/_{imf}, \nu_p)$ are obtained, where $p : N \to N/_{imf}$ is the canonical projection.

c) If $\{(M_i, \mu_i)\}_{i\in I}$ is a family of fuzzy modules, then we can define the product of this families by $\left(\prod_{i\in I} M_i, \mu_A\right)$, where $A = \left\{\pi_i : \prod_{i\in I} M_i \to M_i\right\}$ are the usual projection maps. Coproduct of this families is $\left(\sum_{i\in I} M_i, \mu^B\right)$, where $B = \left\{j_i : M_i \to \sum_{i\in I} M_i\right\}$ are the usual injections.

The following theorem is easily proved.

Theorem 1.1.6. The category of fuzzy modules has zero objects, sums, products, kernels and cokernels.

Lemma 1.1.7. For any module M, the set of grade functions $s(M) = \{\mu : M \to [0,1] | (M, \mu) \text{ is fuzzy module}\}$ is a complete lattice under the following order relation: $\mu \le \eta$ if $\mu(x) \le \eta(x)$ for all $x \in M$.

Proof. Given $\{\mu_i | i \in I\} \subset s(M)$,

$$\underset{i\in I}{\wedge}\{\mu_i\}(x) = \inf_{i\in I}\{\mu_i(x)\}$$

and

$$\underset{i\in I}{\vee}\{\mu_i\}(x) = \inf_{i\in I}\{\delta(x) | \delta \in s(M) \text{ and } \mu_i \le \delta \text{ for all } i \in I\}$$

are respectively the inf and sup of the collection $\{\mu_i | i \in I\}$.

Definition 1.1.8. (i) Let $\Im$ be an arbitrary category. If $S : \Im \to \Im - lat$ is a contravariant functor into the category $\Im - lat$ of complete lattices and order preserving functions such that for each morphism $f : A \to B$ in $\Im$, $S(f) : S(B) \to S(A)$ preserves infima, then S is called a topological theory in $\Im$.

(ii) The top category $\Im^S$ over $\Im$ induced by the topological theory S is defined as follows:

The objects of $\Im^S$ are all x_A with $A \in Ob\Im$ and $x \in S(A)$. A morphism $\widetilde{f} : x_A \to y_B$ is a morphism $f : A \to B$ in $\Im$ satisf ying $x \leq (Sf)(y)$. Composition in $\Im^S$ coincides with composition in $\Im$.

Theorem 1.1.9. $Fz-Mod$ is a top category over modules category.

Prof. It suffices to show that assigning to each module M the corresponding complete lattice $s(M)$ defined in in Lemma 1.1.7 and to every homomorphism $f : M \to N$ the function $s(f) : s(N) \to s(M)$ given by $s(f)(\eta) = \eta_f$ for all $\eta \in s(N)$ determines a contravariant functor $s : Mod \to \Im$. In other words, we need to show that

1) for all $f : M \to N$, $s(f)$ preserves infima,
2) for all $f : M \to N$, $g : N \to T$ we have $s(g \circ f) = s(f) \circ s(g)$, and
3) $s(1) : s(M) \to s(M)$ is the identity function for each identity homomorphism $1 : M \to M$.

In order to prove (1), notice first that the grade function 1 is the infimum of the empty subset of $s(M)[s(N)]$ and indeed $s(f)(1) = 1_f = 1$. Furthermore, for a nonempty family

$\{\eta_i | i \in I\} \subset s(N)$ and $x \in M$,

$$\begin{aligned} s(f)[\wedge\{\eta_i\}](x) &= [\wedge\{\eta_i\}]_f(x) \\ &= [\wedge\{\eta_i\}](f(x)) = \inf\{\eta_i(f(x))\} \\ &= \inf\{(\eta_i)_f(x)\} = [\wedge\{(\eta_i)_f\}](x) \\ &= [\wedge\{s(f)(\eta_i)\}](x). \end{aligned}$$

Hence $s(f)$ preserves infima.

Let $f : M \to N$ and $g : N \to T$ be homomorphism, and let $\tau \in s(T)$. If $x \in M$ then

$$
\begin{aligned}
s(g\circ f)(\tau)(x) &= \tau_{(g\circ f)}(x)\\
&= \tau[(g\circ f)(x)] = \tau(g(f(x)))\\
&= \tau_g(f(x)) = (\tau_g)_f(x)\\
&= [s(f)(\tau_g)](x) = [s(f)(s(g)(\tau))](x)\\
&= [s(f)\circ s(g)](\tau)(x).
\end{aligned}
$$

Therefore $s(g\circ f) = s(f)\circ s(g)$, proving (2).

Lastly, for $\mu \in s(M)$, $x\in M$, if $1: M\to M$ is the identity homomorphism, then $\mu_1(x) = \mu(1(x)) = \mu(x)$, i.e. $s(1)(\mu)=\mu$, and therefore $s(1) = 1 : s(M)\to s(M)$.

By $S(M)(S(z^M))$ and *fz*-sets (*fz*-Ab) we mean the set of all fuzzy subset (fuzzy subgroup) over M and the category of fuzzy subset (fuzzy Abelian subgroups) respectively.

For any infima-preserving function $f: L_1 \to L_2$ in $\Im - lat$, there exist a unique suprema preserving function $g: L_2\to L_1$ such that for

$$a\in L_1,\ b\in L_2,\ f(a)\ge b \Leftrightarrow a\ge g(b).$$

This yields that top categories may be described in terms of a covariant functor $t:\Im\to\Im - lat$ such that for any object A in $\Im$ $t(A)=s(A)$ and for any morphism $f:A\to B$ in $\Im$ the map $t(f): t(A)\to t(B)$ satisfies that for all $\alpha\in t(A)$ and $\beta\in t(B)$, $\beta \ge t(f)(\alpha)$.

Given the right R-module M and left R-module N. We may define a category $I = I_{M,N}$ as follows:

The objects of I are middle linear maps $\sigma: M\times N\to A$, where A is any Abelian group. Given two objects $\sigma_1 : M\times N\to A_1$ and $\sigma_2 : M\times N \to A_2$ in I, a morphism $\theta:\sigma_1\to\sigma_2$ is a group homomorphism $\theta: A_1\to A_2$ such that $\theta\sigma_1 = \sigma_2$.

Definition 1.1.10. Let $f: M \times N \to A$ be a middle linear map and $(A,\alpha) \in fz-Ab$. If (M,μ) is fuzzy right and (N,ν) is fuzzy left R-modules, then $\mu \times \nu : M \times N \to [0,1]$ is defined by

$$(\mu \times \nu)(m,n) = \min\{\mu(m), \nu(n)\}.$$

If $\widetilde{f}:(M \times N, \mu \times \nu) \to (A,\alpha)$ is a morphism in *fz*-sets, then we say that $\widetilde{f}$ is a fuzzy middle linear map.

Now let $fz-I$ be a category having all fuzzy middle linear maps, where for two objects $\sigma_1 :(M \times N, \mu \times \nu) \to (A,\alpha)$ and $\sigma_2 :(M \times N, \mu \times \nu) \to (B,\beta)$ morphism $\theta : \sigma_1 \to \sigma_2$ is a fuzzy group homomorphism $\theta :(A,\alpha) \to (B,\beta)$ such that $\theta\sigma_1 = \sigma_2$.

Lemma 1.1.11. $fz-I$ is a top category on I.

Proof. Define the functor $s: I \to \Im - lat$ as follows:

$s(\sigma : M \times N \to A) = \{\alpha \in s(z^A) | \sigma :(M \times N, \mu \times \nu) \to (A,\alpha) \text{ is a morphism in } fz-sets\}$.

In other words:

$$s(\sigma : M \times N \to A) = \{\alpha \in s(z^A) | \sigma^{-1}\alpha \geq \mu \times \nu)\}$$
$$= \{\alpha \in s(z^A) | \alpha \geq t(\sigma)(\mu \times \nu)\}.$$

Definition 1.1.12. $p: M \times N \to M \otimes N$ be the canonical middle linear map. If (M,μ) is fuzzy right and (N,ν) is fuzzy left R-modules, then the fuzzy tensor product of (M,μ) and (N,ν) is defined by

$$\mu \otimes \nu = \inf\{\beta \in s(z^{M \otimes N} | \beta \geq t(p)(\mu \times \nu)\}$$
$$= \inf\{\beta \in s(z^{M \otimes N} | \mu \times \nu \leq p^{-1}(\beta)\}.$$