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The General Problem of the Motion of Coupled Rigid Bodies about a Fixed Point

Springer Tracts in Natural Philosophy
Volume 7

Springer Tracts in Natural Philosophy

Volume 7

To the Memory of my Parents

Preface

In the theory of motion of several coupled rigid bodies about a fixed point one can distinguish three basic ramifications.

1. The first, the so-called classical direction of investigations, is concerned with particular cases of integrability of the equations of motion of a single rigid body about a fixed point,[1] and with their geometrical interpretation. This path of thought was predominant until the beginning of the 20th century and its most illustrious representatives are L. EULER (1707—1783), J. L. LAGRANGE (1736—1813), L. POINSOT (1777—1859), S. V. KOVALEVSKAYA (1850—1891), and others. Chapter I of the present monograph intends to reflect this branch of investigations.

For collateral reading on the general questions dealt with in this chapter the reader is referred to the following textbooks and reports: A. DOMOGAROV [1], F. KLEIN and A. SOMMERFELD [1_1, 1_2, 1_3], A. G. GREENHILL [10], A. GRAY [1], R. GRAMMEL [4_1], E. J. ROUTH [2_1, 2_2, 3_1, 3_2], J. B. SCARBOROUGH [1], and V. V. GOLUBEV [1, 2].

Chapter II is concerned with the motion of a symmetric as well as an asymmetric self-excited rigid body. A body is said to be self-excited if the torque applied is fixed in the body or moves in a prescribed manner. Prior to the modern age of jet propulsion such a problem seemed to lack a physical meaning. Today, however, various devices with internal reactions are commonly used to influence rotational motions; for example, devices for the steering of space vehicles. Therefore the problem of motion of a self-excited rigid body about a fixed point is now meaningful and important.

Chapter III considers the motion of an externally excited rigid body. While the earlier literature on the motion of rigid bodies is mainly concerned with torque-free and heavy bodies, in the more recent literature several authors such as W. BRAUNBEK [1] and

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1 Any rotating body having freedom in one or more planes at right angles to the plane of rotation is called a gyro or gyroscope. A gyro having complete freedom in three planes at right angles to each other is called a free gyro. Mechanically complete freedom of a wheel in three planes can be realized by mounting it in a system of gimbals. However, a rotating ball held in the air would also be a gyroscope; in fact, the Earth itself is a gyroscope.

R. WIEBELITZ [1] have discussed the motion of a rigid body subject to periodic torque vectors. Such rotating bodies are of interest in astronomy and atomic physics. In astronomy our main concern is the perturbation of the Earth's rotation about its axis under the influence of forces arising from the planetary system (precession and nutation of the Earth's axis). For additional reading on this subject the reader is referred to a report by E. W. WOOLARD [1] and papers by S. D. POISSON [2]. In atomic physics electrons and nuclei in high frequency magnetic fields represent atomic gyroscopes subject to periodic torques. Investigations of this type have been stimulated by the necessity of finding a mechanical model for the phenomenon of nuclear induction. For additional information see F. BLOCH, W. W. HANSEN and M. PACKARD [1], F. BLOCH and A. SIEGERT [1], R. K. WANGSNESS and F. BLOCH [1], and F. KIRCHNER [1].

2. A turn in the direction of research took place in the second decade of the present century when in 1909 the gyroscopic compass and in 1917 the gyro horizon and rate gyro (turn indicator) were constructed. Since the above dates these instruments have been used for guidance and control of ships and aircraft, and today for guidance and control of missiles and spacecraft. As a consequence the applied theory of gyroscopes came into existence and matured rapidly while the classical theory receded into the past. Efforts were now made to investigate the motion and stability of particular gyroscopic devices, to study the effect of motion of the supporting member of such devices, the effect of friction at the bearings and that of the flexibility of the rotor-shaft, and so on. This was the beginning of the second ramification in the general theory of the motion of coupled rigid bodies about a fixed point. The most illustrious representatives of this path of thought are L. FOUCAULT (1819—1878), A. N. KRYLOV (1863—1945), M. SCHULER (1882), R. GRAMMEL (1889—1964), C. S. DRAPER (1901), and others.

For additional reading we refer the reader to the following textbooks and reports: E. S. FERRY [1], R. GRAMMEL [4_2], A. N. KRYLOV and YU. A. KRUTKOV [1], A. L. RAWLINGS [1], M. DAVIDSON [1], K. I. T. RICHARDSON [1], C. S. DRAPER, W. WRIGLEY, and L. R. GROHE [1], E. J. SIFF and C. L. EMMERICH [1], B. V. BULGAKOV [3, 4], R. N. ARNOLD and L. MAUNDER [1], P. SAVET [1], and H. ZIEGLER [2]. Part 4 of F. KLEIN and A. SOMMERFELD's treatise [1] on technical applications of the gyroscope is now rather out of date.

3. A third ramification in the theory of motion of coupled rigid bodies about a fixed point was initiated by Lord KELVIN (Sir William THOMSON) and P. G. TAIT [1]. In the eighties of the last century they were concerned with the classification of the various types of forces. According to their terminology, forces which depend on the generalized

velocities $\dot{q}_k$, and the work of which in any real infinitesimal displacement of the system is equal to zero, are called gyroscopic forces. It should be noted, however, that this term has a conditional meaning in the sense that it applies to actual forces applied to the system as well as to certain terms in the equations of motion which are liable to be interpreted as forces. If the gyroscopic forces are denoted by $g_{ik}\dot{q}_k$ (summation over k), then the matrix of the coefficients g_{ik} (depending upon the coordinates q_k) must be skew-symmetric.

Linear terms with respect to the generalized velocities $\dot{q}_k$ with a skew-symmetric matrix appear, for example, in the nonlinear equations of motion of systems containing gyroscopes as well as in the equations of motion of holonomic systems with cyclical coordinates, or moving subject to nonstationary constraints. Further, they appear in the equations for the perturbations of systems subject to stationary constraints, and in the equations of motion of nonholonomic systems in terms of quasi-coordinates. Hence a general theory of systems moving subject to gyroscopic forces is of interest not only for gyrodynamics but also for various types of mechanical and electrical systems containing no gyroscopes at all.

Investigating the effect of gyroscopic forces on the motion of a given system, it is sometimes convenient to assume that the gyroscopic forces depend on a certain parameter H. The introduction of such a parameter H permits us to study the solutions of the equations of motion as functions of this parameter, and to determine some properties of the system in terms of H. For large values of the parameter H it is natural to raise the question concerning a possible simplification of the equations of motion in order to make the integration easier.

The physical reason for introducing a parameter H is also obvious. Namely, for large values of the velocity $\dot{\Phi}$ of the proper rotation of a gyroscope (in comparison with the velocities of precession $\dot{\psi}$ and nutation $\dot{\Theta}$) $H = Cn$ (Cn being the constant of integration corresponding to the cyclic coordinate Φ in the equations ot motion of a gyroscope) is approximately equal to the modulus I of the angular momentum $\vec{I}$ of the gyroscope, i.e. $I \approx H = Cn$, and the gyroscope exhibits certain properties which do not occur for small values of $\dot{\Phi}$. This simple example shows the necessity and desirability for investigating gyroscopic systems in terms of a parameter.

It is also well known that in certain cases, discarding terms containing second derivatives and products of first derivatives, equations result which provide acceptable solutions for practical purposes. For example, the equations of motion of a gyroscopic compass admit such a simplification and the solution of the simplified equations describes with great accuracy the actual motion. In other cases, however, such a simplification of the differential equations of motion leads to completely unacceptable solutions. For example, the equations of motion of the FOUCAULT gyroscope, being a system with two degrees of freedom, do not admit such a simplification.

A book which summarizes the results obtained following the third path of thought sketched above is that by D. R. MERKIN [1]. It also contains certain general theorems due to its author concerning the stability of the equilibrium of a system subject to gyroscopic forces and the effect of gyroscopic forces on the motion of a nonconservative system. Also of some practical importance are MERKIN's investigations concerning the conditions under which the equations of a fast rotating gyroscope can be simplified in the sense explained above.

Concerning problems of gyroscopic stabilization the reader is referred to I. I. METELICYN [1].

Two particular cases of several-body systems which are of great practical importance are discussed in Chapters IV and V respectively, namely the gyrostat and the gyroscope in a CARDAN suspension.

4. In connection with the investigation of the motion of the Earth's artificial satellites some old problems of classical celestial mechanics and gyrodynamics became again actual and, in addition, new problems arose. Such problems are, to name just a few:

(i) Separation of the general motion of mutually attracting rigid bodies into translations of their mass centers and rotations about the latter.

(ii) Rotation of an artificial satellite about its mass center.

(iii) Motion of a rigid body about a fixed point in a central NEWTONian force field.

(iv) Motion of a rigid body with fluid-filled cavities about a fixed point.

(v) Motion of a gyroscope with variable mass or moments of inertia.

(vi) Application of gyroscopics to inertial guidance systems.

A rapidly expanding literature of the above problems exists today which is already so extensive that a proper survey would be an undertaking beyond the scope of this monograph. Therefore we shall limit ourselves to the discussion of problem (iii) in Chapter VI and problems (i) and (ii) in Chapter VII.

5. The time EULER spent in Berlin (1741—1766) was rich with discoveries in the fields of celestial mechanics, mechanics of rigid bodies and mechanics of fluids. Although EULER was concerned with the dynamics of rigid bodies from the beginning of his scientific activities his main results, which culminated in the now classical equations of

motion of a rigid body about a fixed point, were obtained by him only during the Berlin period of his life and published in the Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin, vols. 5—16 (1751—1767), the papers being written between 1749—1760.

In particular, in 1758 EULER developed the theory of moments of inertia of a rigid body, proved the existence of three mutually orthogonal axes, called the principal axes of inertia of the body, and obtained the equations for the rotational motion of the body relative to the body-fixed coordinate trihedral, the axes of which are directed along the principal inertia axes of the body. The concept of the ellipsoid of inertia, however, was introduced later by L. POINSOT. The above papers of 1758 were published in 1765 together with his fundamental treatise "Theoria Motus Corporum Solidorum seu Rigidorum", of which the 10th and 15th chapters again contained the derivation of his equations.

My report [1] "On some recent advances in the dynamics of rigid bodies and celestial mechanics" appeared in 1958. The present monograph is an attempt not only to account for the present state of the field which it covers but also of its growth during the last two hundred years. Completeness of the monograph is not claimed.

Acknowledgements. This work was supported in part by the Mathematics Division of the United States Air Force Office of Scientific Research (Grant AFOSR 483—64) while the author was on leave from the University of British Columbia.

My colleague Dr. W. H. SIMONS and former student Dr. R. LEE read the manuscript and made some valuable comments. Dr. R. LEE also prepared the figures, and Mrs. Cathie EASTO typed the manuscript. My Ph. D. students Graham ZELMER, B. Sc. (Hons. Math.), B. Sc. (Eng.) and B. S. LALLI, B. A. (Hons.), M. A., and graduate student L. Barry MULLETT, B. Sc., helped me with proof-reading.

To my University, the above agency, J. Springer-Verlag and its technical staff, and all the people named above I wish to express my deep appreciation.

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July 15, 1965
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PART I

Single rigid body

Chapter I

Heavy rigid body

A. General solution of the Euler and Poisson equations

§ 1. The Euler angles

1.1. Definition. According to a theorem of EULER [1] the general displacement of a rigid body with one fixed point is a rotation about some axis through this point. The position of a rigid body is completely determined by locating a rectangular coordinate trihedral fixed in the rigid body relative to a rectangular coordinate trihedral fixed in space. If the fixed point is taken as a common origin of the body and space trihedrals, then the orientation of the body in space can be described in terms of the direction cosines of the body axes relative to the space axes. Among the nine direction cosines only three are independent. Therefore we must use some set of three independent functions of the direction cosines to specify the position of the rigid body. A number of such sets of independent variables have been described in the literature, the most important and useful being the EULER [2] angles.

Let O be a fixed point of a rigid body about which the rotation of the body takes place, and let $OXYZ$ be a right-hand rectangular trihedral fixed in space (Fig. 1). Let $Oxyz$ be a right-hand rectangular trihedral fixed in the body and moving with it. Furthermore, let the coordinate planes XOY and xOy intersect along the line of nodes ON which is perpendicular to the plane through the axes OZ and Oz. Choose the orientation along ON in such a way that the trihedral $ONZz$ is right-handed. Denote the angles ZOz, XON and NOx by Θ, ψ, Φ respectively. These are known as the EULER angles. The angle Θ $(0 \leq \Theta < \pi)$ is called the angle of nutation, the angle ψ $(0 \leq \psi < 2\pi)$

the angle of precession and the angle Φ $(0 \leqq \Phi < 2\pi)$ the angle of proper rotation. If the EULER angles Θ, ψ, Φ are known as functions of the time t, then the position of the trihedral $O\,x\,y\,z$ with respect to the trihedral $O\,X\,Y\,Z$ is defined. In other words, the motion of the rigid body about the fixed point is known.

1.2. The direction cosines of $O\,x$, $O\,y$, $O\,z$ as functions of the Euler angles. The trihedral $O\,x\,y\,z$, the position of which is defined by the EULER angles Θ, ψ, Φ, can be derived from the trihedral $O\,X\,Y\,Z$ by means of the following three successive counter-clockwise rotations: (i) a rotation through the angle ψ about the $O\,Z$-axis, obtaining the trihedral $O\,X_1\,Y_1\,Z_1$, the $O\,X_1$-axis of which coincides with the line of nodes $O\,N$; (ii) a rotation through the angle Θ about the $O\,X_1$-axis, obtaining the trihedral $O\,x_1\,y_1\,z_1$, the $O\,y_1$-axis of which lies in the plane $Z_1\,O\,z_1$ and makes the angle Θ with the $O\,Y_1$-axis; (iii) a rotation through the angle Φ about the $O\,z_1$-axis until the $O\,x_1$-axis coincides with the $O\,x$-axis and the $O\,y_1$-axis with the $O\,y$-axis.

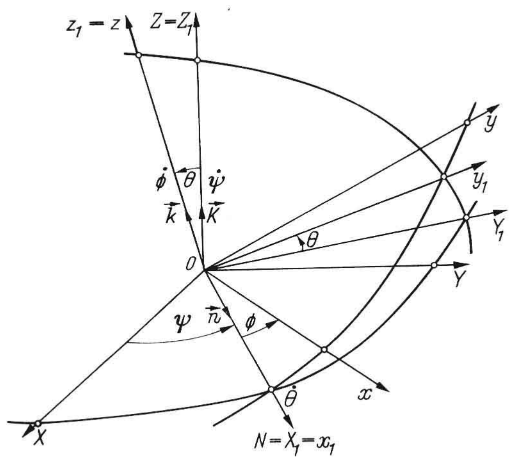

Fig. 1

The transition from coordinates fixed in the rigid body, x, y, z, to coordinates fixed in space, X, Y, Z, can be accomplished by means of an orthogonal matrix A with elements α_{ij} $(i, j = 1, 2, 3)$ connected by six orthogonality conditions

$$\sum_{i=1}^{3} \alpha_{ij}\,\alpha_{ik} = \delta_{jk} \quad \text{or} \quad \sum_{i=1}^{3} \alpha_{ji}\,\alpha_{ki} = \delta_{jk} \qquad (j, k = 1, 2, 3)$$

where δ_{jk} is the KRONECKER δ-symbol, defined by

$$\delta_{jk} = 1\,(j = k), \qquad \delta_{jk} = 0\,(j \neq k)$$

The elements of the resulting transformation matrix A can be obtained by writing the matrix as a triple product of the above three rotations, each of which has a relatively simple matrix form:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} \tag{1.1}$$

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta \\ 0 & \sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \tag{1.2}$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \cos\Phi & -\sin\Phi & 0 \\ \sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.3}$$

Hence the transformation from body coordinates x, y, z to space coordinates X, Y, Z is given by the formulas

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.4}$$

where

$$A = \begin{pmatrix} \cos\psi\cos\Phi - \cos\Theta\sin\psi\sin\Phi, & -\cos\psi\sin\Phi - \cos\Theta\sin\psi\cos\Phi, & \sin\Theta\sin\psi \\ \sin\psi\cos\Phi + \cos\Theta\cos\psi\sin\Phi, & -\sin\psi\sin\Phi + \cos\Theta\cos\psi\cos\Phi, & -\sin\Theta\cos\psi \\ \sin\Theta\sin\Phi & \sin\Theta\cos\Phi & \cos\Theta \end{pmatrix} \tag{1.5}$$

The inverse transformation from space coordinates X, Y, Z to body coordinates x, y, z is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{1.6}$$

where the matrix A^{-1} is equal to the transpose A' of A, i.e.

$$A^{-1} = \begin{pmatrix} \cos\psi\cos\Phi - \cos\Theta\sin\psi\sin\Phi, & \sin\psi\cos\Phi + \cos\Theta\cos\psi\sin\Phi, & \sin\Theta\sin\Phi \\ -\cos\psi\sin\Phi - \cos\Theta\sin\psi\cos\Phi, & -\sin\psi\sin\Phi + \cos\Theta\cos\psi\cos\Phi, & \sin\Theta\cos\Phi \\ \sin\Theta\sin\psi & -\sin\Theta\cos\psi & \cos\Theta \end{pmatrix} \tag{1.7}$$

1.3. The components of the angular velocity $\vec{\omega}$ as functions of the Euler angles. Consider two positions of the rigid body, determined by the EULER angles Θ, ψ, Φ and $\Theta + d\Theta$, $\psi + d\psi$, $\Phi + d\Phi$. The increment $d\Theta$ corresponds to an infinitesimal rotation about the line of nodes ON. Similarly $d\psi$ and $d\Phi$ correspond to infinitesimal rotations about the axes OZ and Oz respectively. Hence the components of the angular velocity $\vec{\omega}$ along the axes $O\,N$, $O\,Z$ and $O\,z$ are $\dot{\Theta}$, $\dot{\psi}$, $\dot{\Phi}$ respectively, and therefore

$$\vec{\omega} = \dot{\Theta}\,\vec{n} + \dot{\psi}\,\vec{K} + \dot{\Phi}\,\vec{k}$$

where $\vec{n}$, $\vec{K}$ and $\vec{k}$ are the unit vectors along the axes $O\,N$, $O\,Z$ and $O\,z$ respectively, and dots denote derivatives with respect to the time t. The table given below is a convenient means of finding the components of $\vec{\omega}$ with respect to the trihedrals $O\,x\,y\,z$ and $O\,X\,Y\,Z$.

		$O\,x$	$O\,y$	$O\,z$	$O\,X$	$O\,Y$	$O\,Z$
	$\vec{\omega}$	p	q	r	P	Q	R
$O\,N$	$\dot{\Theta}$	$\cos\Phi$	$-\sin\Phi$	0	$\cos\psi$	$\sin\psi$	0
$O\,Z$	$\dot{\psi}$	$\alpha_{31} = \alpha$	$\alpha_{32} = \beta$	$\alpha_{33} = \gamma$	0	0	1
$O\,z$	$\dot{\Phi}$	0	0	1	α_{13}	α_{23}	α_{33}

By means of this table we find that

$$\begin{aligned} p &= \dot{\Theta}\cos\Phi + \dot{\psi}\,\alpha \\ q &= -\dot{\Theta}\sin\Phi + \dot{\psi}\,\beta \\ r &= \dot{\psi}\,\gamma + \dot{\Phi} \end{aligned}$$

and

$$\begin{aligned} P &= \dot{\Theta}\cos\psi + \dot{\Phi}\,\alpha_{13} \\ Q &= \dot{\Theta}\sin\psi + \dot{\Phi}\,\alpha_{23} \\ R &= \dot{\Phi}\,\alpha_{33} + \dot{\psi} \end{aligned}$$

Substituting for α, β, γ, and α_{13}, α_{23}, α_{33} their expressions in terms of the Euler angles [see matrix (1.5)], we obtain that

$$\left.\begin{aligned} p &= \dot{\Theta}\cos\Phi + \dot{\psi}\sin\Theta\sin\Phi \\ q &= -\dot{\Theta}\sin\Phi + \dot{\psi}\sin\Theta\cos\Phi \\ r &= \dot{\psi}\cos\Theta + \dot{\Phi} \end{aligned}\right\} \tag{1.8}$$

and

$$\left.\begin{aligned} P &= \dot{\Theta}\cos\psi + \dot{\Phi}\sin\Theta\sin\psi \\ Q &= \dot{\Theta}\sin\psi - \dot{\Phi}\sin\Theta\cos\psi \\ R &= \dot{\Phi}\cos\Theta + \dot{\psi} \end{aligned}\right\} \tag{1.9}$$

§ 2. The Euler and Poisson equations of motion

2.1. The dynamical equations of Euler. According to the basic equation of dynamics the derivative with respect to the time of the angular momentum of a rigid body is equal to the moment of the external