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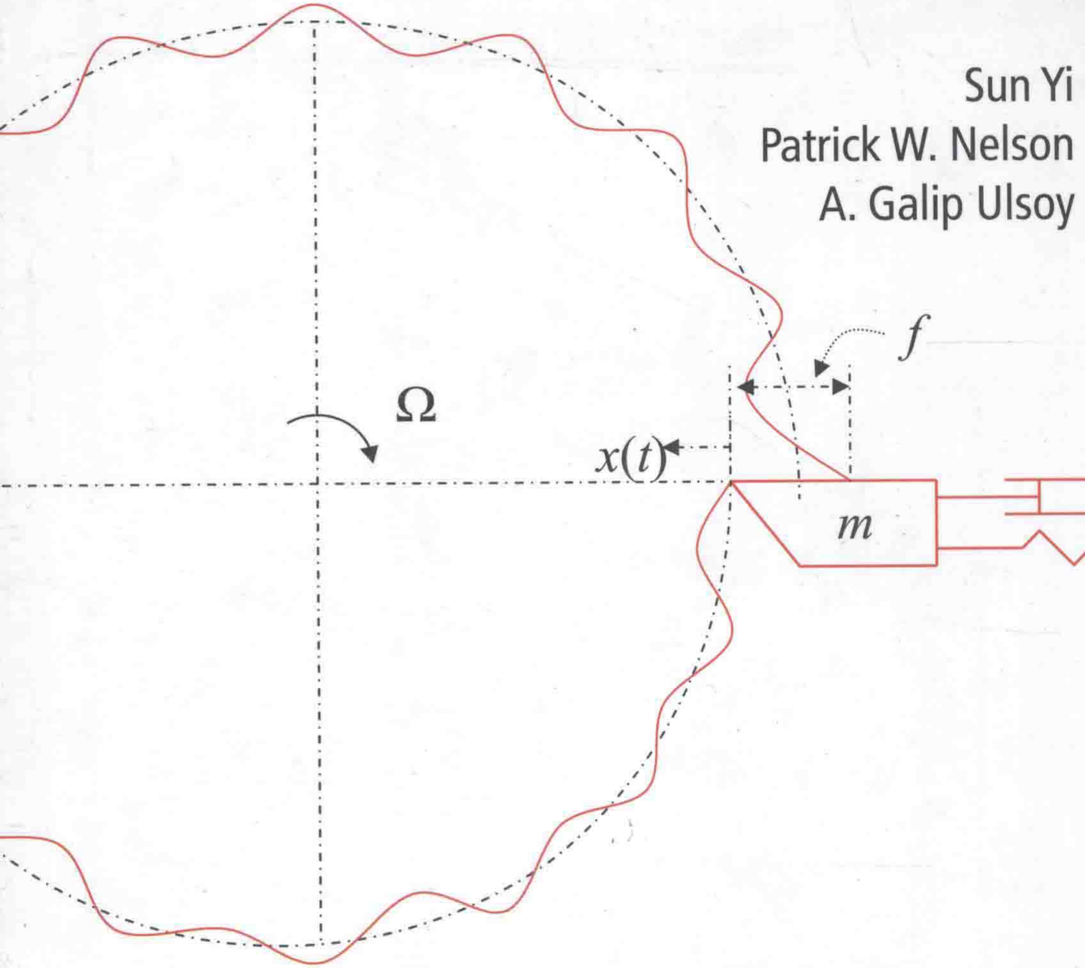

Analysis and Control Using the Lambert W Function

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To our families, with love

Preface

This book is a collection of recent research work on the development of an analytical approach for solutions of delay differential equations via the Lambert W function. It, also, includes methods for analysis and control based on the solutions, and their applications to mechanical and biological systems.

Delay differential equations represent systems that include inherent time-delays in the system or a deliberate introduction of time-delays for control purposes. Such time-delays, common in systems in engineering and science, can cause some significant problems (e.g., instability and inaccuracy) and, thus, limit and degrade the achievable performance of controlled systems. However, due to their innate complexity including infinite-dimensionality, it is not feasible to analyze such systems with classical methods developed for ordinary differential equations (ODEs).

The research presented in this book uses the Lambert W function to obtain free and forced closed-form solutions to such systems. Hence, it provides a more analytical and effective way to treat time-delay systems. The advantage of this approach stems from the fact that the closed-form solution is an infinite series expressed in terms of the parameters of the system. Thus, one can explicitly determine how the parameters are involved in the solution. Furthermore, one can determine how each system parameter affects the eigenvalues of the system. Also, each eigenvalue in the infinite eigenspectrum is associated individually with a branch of the Lambert W function.

The Lambert W function-based approach for the analytical solution to systems of delay differential equations (DDEs) had previously been developed for homogeneous first-order scalar and some special cases of systems of delay differential equations using the Lambert W function as introduced in Chapter 1. In Chapter 2, the analytical solution is extended to the more general case where the coefficient matrices do not necessarily commute, and to the nonhomogeneous case. The solution is in the form of an infinite series of modes written in terms of the matrix Lambert W function. The derived solution is used to investigate the stability of time-delay systems via dominant eigenvalues in terms of the Lambert W function. It is also applied to the regenerative machine tool chatter problem of a manufacturing process in Chapter 3. Based on the solution form in terms of the matrix Lambert W function, algebraic conditions and Gramians for controllability and observability of DDEs are derived in a manner analogous to the well-known controllability and observability results for ODEs in Chapter 4. In Chapter 5, the problem of feedback controller design via eigenvalue assignment for linear time-invariant time-delay systems is considered. The method for eigenvalue assignment is extended to design robust controllers for time-delay systems with uncertainty and to improve transient response in Chapter 6. For systems where all state variables cannot be measured directly, a new approach for observer-based feedback control is developed and applied to diesel engine control in Chapter 7. In Chapter 8, the approach using the Lambert W function is applied to analyze a HIV pathogenesis dynamic model with an intracellular delay.

The authors hope that this book will be of interest to graduate students and researchers in engineering and mathematics who have special interest in studying the properties, and in designing controllers, for time-delay systems.

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Contents

Chapter 1

Introduction

1.1 Motivation

Time-delay systems (TDS) arise from inherent time-delays in the components of the systems, or from the deliberate introduction of time-delays into the systems for control purposes. Such time-delays occur often in systems in engineering, biology, chemistry, physics, and ecology (Niculescu, 2001). Time-delay systems can be represented by delay differential equations (DDEs), which belong to the class of functional differential equations, and have been extensively studied over the past decades (Richard, 2003). Such time-delays can limit and degrade the achievable performance of controlled systems, and even induce instability. Time-delay terms lead to an infinite number of roots of the characteristic equation, making systems difficult to analyze with classical methods, especially, in checking stability and designing stabilizing controllers. Thus, such problems are often solved indirectly by using approximations. A widely used approximation method is the Padé approximation, which is a rational approximation and results in a shortened fraction as a substitute for the exponential time-delay term in the characteristic equation. However, such an approach constitutes a limitation in accuracy, can lead to instability of the actual system and induce non-minimum phase and, thus, high-gain problems (Silva and Datta, 2001). Prediction-based methods (e.g., Smith predictor (Smith, 1957), finite spectrum assignment (FSA) (Zhong, 2006), and adaptive Posicast (Niculescu and Annaswamy, 2003)) have been used to stabilize time-delay systems by transforming the problem into a non-delay system. Such methods require model-based calculations, which may cause unexpected errors when applied to a real system. Furthermore, safe implementation of such methods is still an open problem due to computational issues. Controllers have also been designed using the Lyapunov framework (e.g., linear matrix inequalities

(LMIs) or algebraic Riccati equations (AREs)) (Gu and Niculescu, 2006; Liu, 2003). These methods require complex formulations, and can lead to conservative results and possibly redundant control.

To find more effective methods, an analytic approach to obtain the complete solution of systems of delay differential equations based on the concept of the Lambert W function, which has been known to be useful to analyze DDEs (Corless *et al.*, 1996), was developed in (Asl and Ulsoy, 2003). The solution has an analytical form expressed in terms of the parameters of the DDE and, thus, one can explicitly determine how the parameters are involved in the solution and, furthermore, how each parameter affects each eigenvalue and the solution. Also, each eigenvalue is associated individually with a particular 'branch' of the Lambert W function. In this book, the analytical approach using the Lambert W function is extended to general systems of DDEs and non-homogeneous DDEs, and compared with the results obtained by numerical integration. The advantage of this approach lies in the fact that the form of the solution obtained is analogous to the general solution form of ordinary differential equations, and the concept of the state transition matrix in ODEs can be generalized to DDEs using the concept of the matrix Lambert W function. This suggests that some approaches for analysis and control used for systems of ODEs, based on concept of the state transition matrix, can potentially be extended to systems of DDEs. These include analysis of stability, controllability and observability, and methods for eigenvalue assignment for linear feedback controller design with an observer, and extension to robust stability and time-domain specifications. Also, the approaches developed based on the proposed solution method are applied to time-delay systems in engineering and biology as discussed in subsequent chapters.

1.2 Background

1.2.1 *Delay differential equation*

Delay differential equations are also known as difference-differential equations, were initially introduced in the 18th century by Laplace and Condorcet (Gorecki *et al.*, 1989). Delay differential equations are a type of differential equation where the time derivatives at the current time depend on the solution, and possibly its derivatives, at previous times. A class of such equations, that involve derivatives with delays as well as the solution itself have historically been called *neutral* DDEs (Hale and Lunel, 1993). In this book only *retarded* DDEs, where there is no time-delay in the derivative terms, are considered.

The basic theory concerning stability and works on fundamental theory, e.g., existence and uniqueness of solutions, was presented in (Bellman and Cooke, 1963). Since then, DDEs have been extensively studied in recent decades and a great number of monographs have been published including significant works on dynamics of DDEs by Hale and Lunel (1993), on stability by Niculescu (2001), and so on. The reader is referred to the detailed review in (Richard, 2003; Gorecki *et al.*, 1989; Hale and Lunel, 1993). The interest in the study of DDEs is caused by the fact that many processes have time-delays and have been modeled for better fidelity by systems of DDEs in the sciences, engineering, economics, etc. (Niculescu, 2001). Such systems, however, are still not feasible to precisely analyze and control, thus, the study of systems of DDEs has actively been conducted during recent decades (Richard, 2003).

1.2.2 *Lambert W function*

Introduced in the 1700s by Lambert and Euler (Corless *et al.*, 1996), the Lambert W function is defined to be any function, $W(H)$, that satisfies

$$W(H)e^{W(H)} = H \tag{1.1}$$

The Lambert W function is complex valued, with a complex argument, H, and has an infinite number of branches, W_k, where $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$ (Asl and Ulsoy, 2003). Figure 1.1 shows the range of each branch of the Lambert W function. For example, the real part of the principal branch, W_0, has a minimum value, -1. The principal and all other branches of the Lambert W function in Eq. (1.1) can be calculated analytically using a series expansion (Corless *et al.*, 1996), or alternatively, using commands already embedded in the various commercial software packages, such as Matlab, Maple, and Mathematica.

An analytic approach to obtain the complete solution of systems of delay differential equations based on the concept of the Lambert W function was developed by Asl and Ulsoy (2003). Consider a first-order scalar homogenous DDE:

$$\begin{aligned} \dot{x}(t) &= ax(t) + a_d x(t-h), & t > 0 \\ x(0) &= x_0, & t = 0 \\ x(t) &= g(t), & t \in [-h, 0) \end{aligned} \tag{1.2}$$

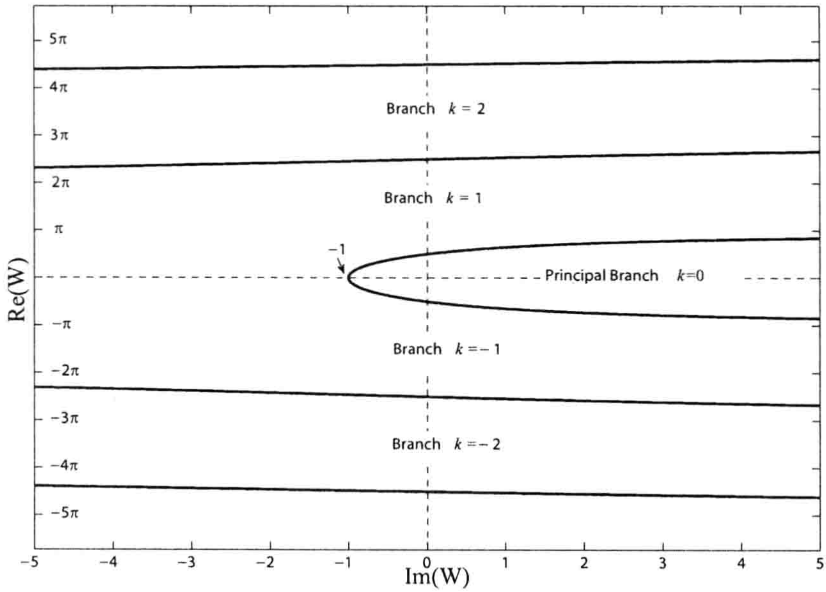

Fig. 1.1 Ranges of each branch of the Lambert W function (Corless *et al.*, 1996). Note that real part of the principal branch, W_0, is equal to or larger than -1.

Instead of a simple initial condition as in ODEs, two initial conditions need to be specified for DDEs: a preshape function, $g(t)$, for $-h \leq t < 0$ and initial point, x_0, at time, $t = 0$. This permits a discontinuity at $t = 0$, when $x_0 \neq g(t = 0)$. The quantity, h, denotes the time-delay. The solution to Eq. (1.2) can be derived in terms of an infinite number of branches of the Lambert W function, defined in Eq. (1.1), (Asl and Ulsoy, 2003):

$$x(t) = \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^I, \quad \text{where } S_k = \frac{1}{h} W_k(a_d h e^{-ah}) + a \tag{1.3}$$

The coefficient, C_k^I, is determined numerically from the preshape function, $g(t)$, and initial state, x_0, defined in the Banach space as described by Asl and Ulsoy (2003). The analytic methods to find the coefficient, C_k^I and the numerical and analytic methods for other coefficients for non-homogeneous and higher order of DDEs are also developed in a subsequent chapter. Note that, unlike results by other existing methods, the solution in Eq. (1.3) has an analytical form expressed in terms of the parameters of the DDE in Eq. (1.2), i.e., a, a_d and h. One can explicitly determine how the parameters are involved in the solution and, furthermore, how each parameter affects each eigenvalue and the solution. Also, each eigenvalue is distinguished by k, which indicates the branch of the Lambert W function as seen in Eq. (1.3).

1.3 Scope of This Document

This book presents the derivation of solutions of systems of DDEs, and the development of methods to analyze and control time-delay systems with application to systems in engineering and biology. This new technique allows one to study how the parameters in time-delay systems are involved in the solution, which is essential to investigate system properties, such as stability, controllability, observability, and sensitivity. Finally, controllers for time-delay systems, with observers, are designed via eigenvalue assignment to improve robust stability and to meet time-domain specifications as well as to stabilize unstable systems (See Fig. 1.2).

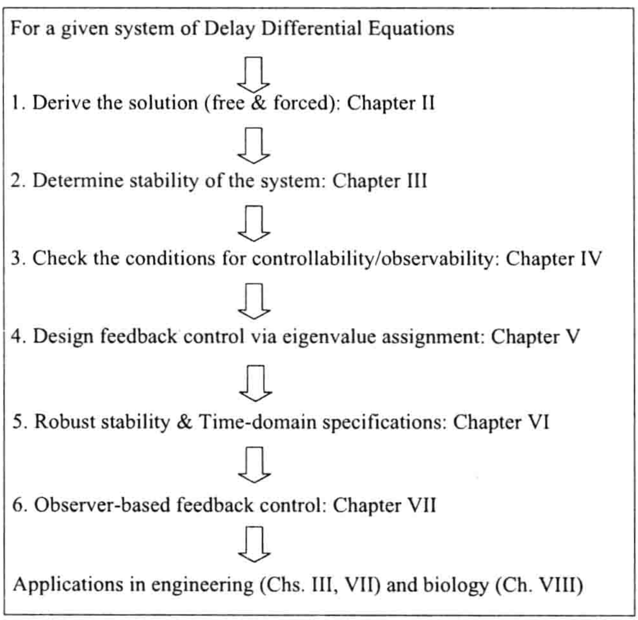

Fig. 1.2 The matrix Lambert W function-based approach: using the approach developed in this research, the steps in the figure, which are standard for systems of ODEs, become tractable for DDEs.

Because each chapter of this book is based on manuscripts that have been published in a journal or conference, the background material for each is included in the relevant chapters. The remaining chapters are summarized as follows.

Chapter II: "Solutions of Systems of DDEs via the Matrix Lambert W Function", which was published in the *Dynamics of Continuous, Discrete and Impulsive Systems (Series A)* (Yi *et al.*, 2007d) and an early version of this work was presented in part at the 2006 American Control Conference (Yi and Ulsoy, 2006) and in part at the 2006 IEEE Conference on Decision and Control (Yi *et al.*, 2006b). Previously, an approach for the analytical solution to systems of DDEs had been developed for homogeneous scalar and some special cases of systems of delay differential equations using the Lambert W function (Asl and Ulsoy, 2003). In this chapter, the approach is extended to the more general case where the coefficient matrices in a system of DDEs do not necessarily commute, and to the nonhomogeneous cases. The solution is in the form of an infinite series of modes written in terms of the *matrix* Lambert W function. The form of the obtained solution has similarity to the concept of the state transition matrix in linear ordinary differential equations, enabling its use for general classes of linear delay differential equations. Examples are presented to illustrate the new approach by comparison to numerical methods. The analytical solution in terms of the Lambert W function is also presented in the Laplace domain to reinforce the analogy to ODEs.

Chapter III: "Stability of Systems of Delay Differential Equations via the Matrix Lambert W Function: Application to machine tool chatter," which was published in the *Mathematical Biosciences and Engineering* (Yi *et al.*, 2007b) and an earlier version of this work was presented at the 2006 ASME International Conference on Manufacturing Science and Engineering (Yi *et al.*, 2006a). This chapter investigates stability of systems of DDEs using the solution derived in terms of the parameters of systems in Chapter II. By applying the matrix Lambert W function-based approach to the chatter equation, one can solve systems of DDEs in the time domain, obtain dominant eigenvalues, and check the stability of the system. With this method one can obtain ranges of preferred operating spindle speed that do not cause chatter to enhance productivity of processes and quality of products. The new approach shows excellent accuracy and certain other advantages, when compared to existing graphical, computational and approximate methods.

Chapter IV: "Controllability and Observability of Systems of Linear Delay Differential Equations via the Matrix Lambert W Function," which was published in the *IEEE Transactions on Automatic Control* (Yi *et al.*, 2008a) and an earlier version of this work was presented at the 2007 American Control Conference (Yi *et al.*, 2007a). Controllability and observability of linear time-delay systems has been studied, and various definitions and criteria have been presented since the 1960s (Malek-Zavarei and Jamshidi, 1987), (Yi *et al.*, 2008a). However, the lack of an analytical solution approach has limited the applicability of the existing theory. In this chapter, based on the solution form in terms of the matrix Lambert W function, algebraic conditions and Gramians for controllability and observability of DDEs were derived in a manner analogous to the well-known controllability and observability results for the ODE case. The controllability and observability Gramians indicate how controllable and observable the corresponding states are, while algebraic conditions tell only whether a system is controllable/observable or not. With the Gramian concepts, one can determine how the changes in some specific parameters of the system affect the controllability and observability of the system via the resulting changes in the Gramians. Furthermore, for systems of ODEs, a balanced realization in which the controllability Gramian and observability Gramian of a system are equal and diagonal was introduced in (Moore, 1981). Using the Gramians defined in this chapter, the concept of the balanced realization has been extended to systems of DDEs for the first time.

Chapter V: "Eigenvalue Assignment via the Lambert W Function for Control for Time-Delay Systems," which is in press in the *Journal of Vibration and Control* (Yi *et al.*, 2010b) and an earlier version of this work was presented at the 2007 ASME International Design Engineering Technical Conferences (Yi *et al.*, 2007c). In this chapter, the problem of feedback controller design via eigenvalue assignment for linear time-invariant systems of linear delay differential equations with a single delay is considered. Unlike ordinary differential equations, DDEs have an infinite eigenspectrum and it is not feasible to assign all closed-loop eigenvalues. However, one can assign a critical subset of them using a solution to linear systems of DDEs in terms of the matrix Lambert W function. The solution has an analytical form expressed in terms of the parameters of the DDE, and is similar to the state transition matrix in linear ODEs. Hence, one can extend controller design methods developed based upon the solution form of systems of ODEs to systems of DDEs, including the design of feedback controllers