Graduate Texts in Mathematics

123

Numbers

Springer-Verlag

H.-D. Ebbinghaus H. Hermes
F. Hirzebruch M. Koecher K. Mainzer
J. Neukirch A. Prestel R. Remmert

Numbers

With an Introduction by K. Lamotke
Translated by H.L.S. Orde
Edited by J.H. Ewing

With 24 Illustrations

Springer-Verlag
New York Berlin Heidelberg London
Paris Tokyo Hong Kong Barcelona

Heinz-Dieter Ebbinghaus
Hans Hermes
Mathematisches Institut
Universität Freiburg
Albertstraße 23b, D-7800
Freiburg, FRG

Friedrich Hirzebruch
Max-Planck-Institut für
Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3, FRG

Klaus Lamotke (*Editor of German Edition*)
Mathematisches Institut
der Universität zu Köln
Weyertal 86–90, D-5000
Köln, FRG

Max Koecher (1924–1990)
Reinhold Remmert
Mathematisches Institut
Universität Münster
Einsteinstraße 62
D-4400 Münster, FRG

Klaus Mainzer
Lehrstuhl für Philosophie und
Wissenschaftstheorie
Universität Augsburg
Universitätsstraße 10
D-8900 Augsburg, FRG

H.L.S. Orde (*Translator*)
Bressenden
Biddenden near Ashford
Kent TN27 8DU, UK

Jürgen Neukirch
Naturwissenschaftliche Fakultät
I Mathematik
Universitätsstraße 31
D-8400 Regensburg, FRG

Alexander Prestel
Fakultät für Mathematik
Universität Konstanz
Postfach 5560, D-7750
Konstanz, FRG

John H. Ewing (*Editor of English Edition*)
Department of Mathematics
Indiana University
Bloomington, IN 47405, USA

Mathematics Subject Classification (1980): 00A05

Library of Congress Cataloging-in-Publication Data
Zahlen. Grundwissen Mathematik 1. English
Numbers / Heinz-Dieter Ebbinghaus . . . [et al.]; with an introduction by Klaus Lamotke; translated by H.L.S. Orde; edited by John H. Ewing.
p. cm.—(Readings in mathematics)
Includes bibliographical references.
ISBN 0-387-97497-0
1. Number theory. I. Ebbinghaus, Heinz-Dieter. II. Ewing, John H. III. Series. Graduate texts in mathematics. Readings in mathematics.
QA241.Z3413 1991
512'.7—dc20 89-48588

This book is a translation of the second edition of *Zahlen*, Grundwissen Mathematik 1, Springer-Verlag, 1988. The present volume is the first softcover edition of the previously published hardcover version (ISBN 0-387-97202-1).

Camera-ready copy prepared using LaTeX.
Printed and bound by R.R. Donnelley & Sons, Harrisonburg, Virginia.
Printed in the United States of America.

9 8 7 6 5 4 3 2 Printed on acid-free paper.

ISBN 0-387-97497-0 Springer-Verlag New York Berlin Heidelberg
ISBN 3-540-97497-0 Springer-Verlag Berlin Heidelberg New York

Preface to the English Edition

A book about numbers sounds rather dull. This one is not. Instead it is a lively story about one thread of mathematics—the concept of "number"—told by eight authors and organized into a historical narrative that leads the reader from ancient Egypt to the late twentieth century. It is a story that begins with some of the simplest ideas of mathematics and ends with some of the most complex. It is a story that mathematicians, both amateur and professional, ought to know.

Why write about numbers? Mathematicians have always found it difficult to develop broad perspective about their subject. While we each view our specialty as having roots in the past, and sometimes having connections to other specialties in the present, we seldom see the panorama of mathematical development over thousands of years. *Numbers* attempts to give that broad perspective, from hieroglyphs to K-theory, from Dedekind cuts to nonstandard analysis. Who first used the standard notation for π (and who made it standard)? Who were the "quaternionists" (and can their zeal for quaternions tell us anything about the recent controversy concerning Chaos)? What happened to the endless supply of "hypercomplex numbers" or to quaternionic function theory? How can the study of maps from projective space to itself give information about algebras? How did mathematicians resurrect the "ghosts of departed quantities" by reintroducing infinitesimals after 200 years? How can games be numbers and numbers be games? This is mathematical culture, but it's not the sort of culture one finds in scholarly tomes; it's lively culture, meant to entertain as well as to inform.

This is not a book for the faint-hearted, however. While it starts with material that every undergraduate could (and should) learn, the reader is progressively challenged as the chapters progress into the twentieth century. The chapters often tell about people and events, but they primarily tell about mathematics. Undergraduates can certainly read large parts of this book, but mastering the material in late chapters requires work, even for mature mathematicians. This is a book that can be read on several levels, by amateurs and professionals alike.

The German edition of this book, *Zahlen,* has been quite successful. There was a temptation to abbreviate the English language translation by making it less complete and more compact. We have instead tried to produce a faithful translation of the entire original, which can serve as a scholarly reference as well as casual reading. For this reason, quotations are included along with translations and references to source material in foreign languages are included along with additional references (usually more recent) in English.

Translations seldom come into the world without some labor pains. Authors and translators never agree completely, especially when there are eight authors and one translator, all of whom speak both languages. My job was to act as referee in questions of language and style, and I did so in a way that likely made neither side happy. I apologize to all.

Finally, I would like to thank my colleague, Max Zorn, for his helpful advice about terminology, especially his insistence on the word "octonions" rather than "octaves."

March 1990 John Ewing

Preface to Second Edition

The welcome which has been given to this book on numbers has pleasantly surprised the authors and the editor. The scepticism which some of us had felt about its concept has been dispelled by the reactions of students, colleagues and reviewers. We are therefore very glad to bring out a second edition—much sooner than had been expected. We have willingly taken up the suggestion of readers to include an additional chapter by J. NEUKIRCH on p-adic numbers. The chapter containing the theorems of FROBENIUS and HOPF has been enlarged to include the GELFAND–MAZUR theorem. We have also carefully revised all the other chapters and made some improvements in many places. In doing so we have been able to take account of many helpful comments made by readers for which we take this opportunity of thanking them. P. ULLRICH of Münster who had already prepared the name and subject indexes for the first edition has again helped us with the preparation of the second edition and deserves our thanks.

Oberwolfach, March 1988 Authors and Publisher

Preface to First Edition

The *basic mathematical knowledge* acquired by every mathematician in the course of his studies develops into a unified whole only through an awareness of the multiplicity of relationships between the individual mathematical theories. Interrelationships between the different mathematical disciplines often reveal themselves by studying historical development. One of the main underlying aims of this series is to make the reader aware that mathematics does not consist of isolated theories, developed side by side, but should be looked upon as an organic whole.

The present book on numbers represents a departure from the other volumes of the series inasmuch as seven authors and an editor have together contributed thirteen chapters. In conversations with one another the authors agreed on their contributions, and the editor endeavored to bring them into harmony by reading the contributions with a critical eye and holding subsequent discussions with the authors. The other volumes of the series can be studied independently of this one.

While it is impossible to name here all those who have helped us by their comments, we should nevertheless like to mention particularly Herr Gericke (of Freiburg) who helped us on many occasions to present the historical development in its true perspective.

K. Peters (at that time with Springer-Verlag) played a vital part in arranging the first meeting between the publisher and the authors. The meetings were made possible by the financial support of the Volkswagen Foundation and Springer-Verlag, as well as by the hospitality of the Mathematical Research Institute in Oberwolfach.

To all of these we extend our gratitude.

Oberwolfach, July 1983 Authors and Editor

Contents

Introduction

K. Lamotke

Mathematics, according to traditional opinion, deals with numbers and figures. In this book we do not begin, as EUCLID began, with figures but with numbers.

Mathematical research over the last hundred years has created abstract theories, such as set theory, general algebra, and topology, whose ideas have now penetrated into the teaching of mathematics at the elementary level. This development has not been ignored by the authors of this book; indeed, they have willingly taken advantage of it in that the authors assume the reader to be familiar with the basic concepts of (naive) set theory and algebra. On the other hand, a first volume on numbers should emphasize the fact that modern research in mathematics and its applications is, to a considerable extent, linked to what was created in the past. In particular, the traditional number system is the most important foundation of all mathematics.

The book that we now present is divided into three parts, of which the first, which may be regarded as the heart, describes the structure of the number-system, from the natural numbers to the complex and p-adic numbers. The second part deals with its further development to 'hypercomplex numbers,' while in the third part two relatively new extensions of the real number system are presented. The six chapters of the first part cover those parts of the subject of 'numbers' that every mathematician ought to have heard or read about at some time. The other two parts are intended to satisfy the appetite of a reader who is curious to learn something beyond the basic facts. On the whole, "the structure of number systems" would be a more accurate description of the content of this book.

We should now like to say a few words in more detail about the various contributions, the aims that the authors have set out to achieve, and the reasons that have induced us to bring them together in the form in which they are presented here.

Part A

Since the end of the last century it has been customary to construct the number system by beginning with the natural numbers and then extending the structure step-by-step to include the integers, the rational numbers, the real numbers, and finally the complex numbers. That is not, however, the way in which the concept of number developed historically. Even in ancient times, the rational numbers (fractions and ratios) and certain irrational numbers (such as π, the ratio of the circumference to the radius of a circle, and square-roots) were known in addition to the natural numbers. The system of (positive) rational and irrational numbers was also described theoretically by Greek philosophers and mathematicians, but it was done within the framework of an autonomous theory of commensurable and incommensurable proportions, and it was not thought of as an extension of the natural numbers. It was not until after many centuries of working numerically with proportions that the realization dawned in the 17th century that a number is something that bears the same relationship to (the unit) one as a line segment bears to another given segment (of unit length). Negative numbers, which can be shown to have been in use in India in the 6th century, and complex numbers, which CARDAN took into consideration in 1545 as a solution of a quadratic equation, were still looked upon as questionable for a long time afterwards. In the course of the 19th century the construction that we use today began to emerge.

Each chapter contains a contribution that includes a description of the historical development of the fundamental concepts. These contributions are not intended to replace a history of the number concept, but are aimed at contributing towards a better understanding of the modern presentation by explaining the historical motivation.

In this sense, Chapter 1, §1 begins with the oldest of the representations of numbers that have been handed down to us by tradition, and leads into §2 in which the ideas involved in counting are given axiomatically following the methods introduced by DEDEKIND, by using the concepts of set-theory.

In the ensuing step-by-step construction of the number-system certain themes constantly recur. (1) The step from one stage to the next is prompted each time by the desire to solve problems that can be formulated but not solved in terms of numbers defined so far. (2) The number system of the next stage is constructed, with the help of the operations of set-theory, as an extension of the existing system designed to make the initial problem solvable. For this the following items are necessary. (3) The existing computational operations and relations must be carried over to the new system. (4) The validity of all the computational rules in the new context has to be checked. The processes (1) to (3) are always carried out, in the chapters that follow, but item (4) usually involves tedious verifications, which soon become a matter of routine. Here the authors allow themselves to carry out