SINGULAR POINTS OF COMPLEX HYPERSURFACES

BY

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Preface

The topology associated with a singular point of a complex curve has fascinated a number of geometers, ever since K. BRAUNER* showed in 1928 that each such singular point can be described in terms of an associated knotted curve in the 3-sphere. Recently E. BRIESKORN has brought new interest to the subject by discovering similar examples in higher dimensions, thus relating algebraic geometry to higher dimensional knot theory and the study of exotic spheres.

This manuscript will study singular points of complex hypersurfaces by introducing a fibration which is associated with each singular point.

As prerequisites the reader should have some knowledge of basic algebra and topology, as presented for example in LANG, *Algebra* or VAN DER WAERDEN, *Modern Algebra,* and in SPANIER, *Algebraic Topology*.

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* See the Bibliography. Proper names in capital letters will always indicate a reference to the Bibliography.

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§1. INTRODUCTION

Let $f(z_1, \ldots, z_{n+1})$ be a non-constant polynomial in $n+1$ complex variables, and let V be the algebraic set consisting of all $(n+1)$-tuples

$$\mathbf{z} = (z_1, \ldots, z_{n+1})$$

of complex numbers with $f(\mathbf{z}) = 0$. (Such a set is called a *complex hypersurface.*) We want to study the topology of V in the neighborhood of some point $\mathbf{z}^0$.

We will use the following construction, due to BRAUNER. Intersect the hypersurface V with a small sphere S_ε centered at the given point $\mathbf{z}^0$. Then the topology of V within the disk bounded by S_ε is closely related to the topology of the set

$$K = V \cap S_\varepsilon .$$

(Compare §2.10 and §2.11.)

As an example, if $\mathbf{z}^0$ is a *regular point* of f (that is if some partial derivative $\partial f/\partial z_j$ does not vanish at $\mathbf{z}^0$) then V is a smooth manifold of real dimension $2n$ near $\mathbf{z}^0$. The intersection K is then a smooth $(2n-1)$-dimensional manifold, diffeomorphic to the $(2n-1)$-sphere, and K is embedded in an unknotted manner in the $(2n+1)$-sphere S_ε. (See §2.12.)

By way of contrast, consider the polynomial

$$f(z_1, z_2) = z_1^p + z_2^q$$

in two variables, with a *critical point* $(\partial f/\partial z_1 = \partial f/\partial z_2 = 0)$ at the origin. Assume that the integers p, q are relatively prime and ≥ 2.

ASSERTION (Brauner). *The intersection of* $V = f^{-1}(0)$ *with a sphere* S_ε *centered at the origin is a knotted circle of the type known as a "torus knot of type* (p, q)*" in the 3-sphere* S_ε.

[*Proof:* It is easily verified that the intersection K lies in the torus consisting of all (z_1, z_2) with $|z_1| = \xi$, $|z_2| = \eta$ where ξ and η are positive constants. In fact, K consists of all pairs $(\xi e^{qi\theta}, \eta e^{pi\theta + \pi i/q})$ as the parameter θ ranges from 0 to 2π: Thus K sweeps around the torus q times in one coordinate direction and p times in the other.]

For example the torus knot of type (2, 3) is illustrated in Figure 1.

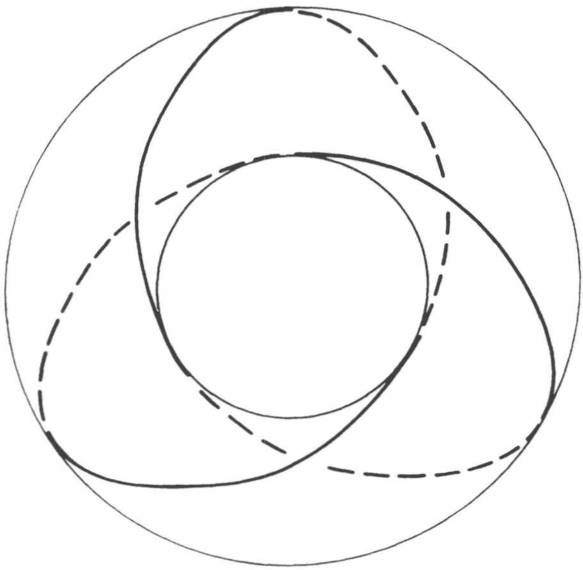

Figure 1. The torus knot of type (2, 3).

(By using more complicated polynomials one can of course arrive at much more complicated knots. Compare §10.11.)

BRIESKORN has studied higher dimensional analogues of these torus knots. For example let $V(3, 2, 2, \ldots, 2)$ be the locus of zeros of the polynomial

$$f(z_1, \ldots, z_{n+1}) = z_1{}^3 + z_2{}^2 + \cdots + z_{n+1}{}^2 .$$

For all odd values of n this hypersurface intersects S_ε in a smooth manifold K which is homeomorphic to the sphere S^{2n-1}. In some cases (for example when $n = 3$) K is diffeomorphic to the standard $(2n-1)$-sphere, while in other cases (for example $n = 5$) K is an "exotic" sphere. But in all cases K is embedded in a knotted manner in the $(2n+1)$-sphere S_ε.

These Brieskorn spheres will be studied in more detail in §9.

The object of this paper is to introduce a fibration which is useful in describing the topology of such intersections

$$K = V \cap S_\varepsilon \subset S_\varepsilon \quad .$$

Here are some of the main results, which will be proved in Sections 4 through 7.

FIBRATION THEOREM. *If* $\mathbf{z}^0$ *is any point of the complex hypersurface* $V = f^{-1}(0)$ *and if* S_ε *is a sufficiently small sphere centered at* $\mathbf{z}^0$, *then the mapping*

$$\phi(\mathbf{z}) = f(\mathbf{z})/|f(\mathbf{z})|$$

from $S_\varepsilon - K$ *to the unit circle is the projection map of a smooth fiber bundle*. Each fiber*

$$F_\theta = \phi^{-1}(e^{i\theta}) \subset S_\varepsilon - K$$

is a smooth parallelizable 2n-*dimensional manifold.*

If the polynomial f has no critical points near $\mathbf{z}^0$, except for $\mathbf{z}^0$ itself, then we can give a much more precise description.

THEOREM. *If* $\mathbf{z}^0$ *is an isolated critical point of* f, *then each fiber* F_θ *has the homotopy type of a bouquet* $S^n \vee \cdots \vee S^n$ *of* n-*spheres, the number of spheres in this bouquet (i.e., the middle Betti number of* F_θ*), being strictly positive. Each fiber can be considered as the interior of a smooth compact manifold-with-boundary,*

$$\text{Closure}\,(F_\theta) = F_\theta \cup K \ ,$$

* The term "fiber bundle" will be used as a synonym for "locally trivial fiber space."

where the common boundary K *is an* $(n-2)$*-connected manifold.*

Thus all of the fibers F_θ fit around their common boundary K in the manner illustrated in Figure 2. The smooth manifold K is connected if $n \geq 2$, and simply connected if $n \geq 3$.

Here is a more detailed outline of what follows. Section 2 describes elementary properties of real algebraic sets, following WHITNEY. A fundamental lemma concerning the existence of real analytic curves on real algebraic sets is proved in §3. All of the subsequent proofs rely on this lemma. The basic fibration theorem is proved in §4. Further details on the topology of K and F_θ are obtained in §5.

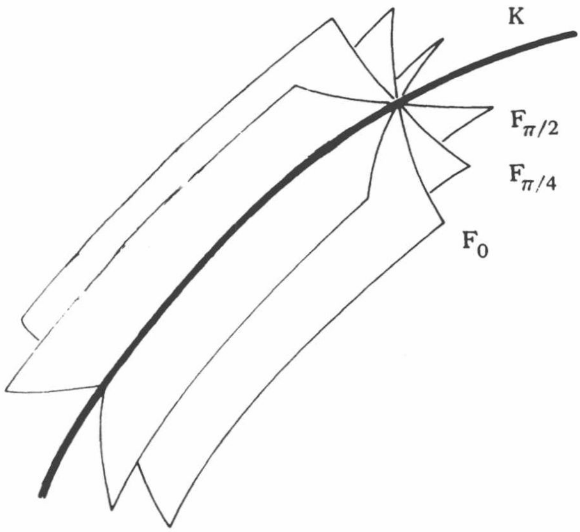

Figure 2.

Next we introduce the additional hypothesis that the origin is an *isolated* critical point of f. Then a much more precise description of the fiber is possible (§6), and a precise formula for the middle Betti number of the fiber is given (§7). The topology of the intersection K is then described in terms of a certain polynomial $\Delta(t)$ with integer coefficients which generalizes the Alexander polynomial of a knot. (§8.)

The Brieskorn examples of singular varieties which are topologically manifolds are described in §9, and the classical theory of singular points of complex curves is described in §10. The last section proves a generalization of the fibration theorem to certain systems of real polynomials. As an example, a polynomial description of the Hopf fibrations is given.

Two appendices conclude the presentation.

§2. ELEMENTARY FACTS ABOUT REAL OR COMPLEX ALGEBRAIC SETS

Let Φ be any infinite field, and let Φ^m be the coordinate space consisting of all m-tuples $x = (x_1, \ldots, x_m)$ of elements of Φ. (We are principally interested in the case where Φ is the field R of real numbers or the field C of complex numbers.)

DEFINITION. A subset $V \subset \Phi^m$ is called an *algebraic set** if V is the locus of common zeros of some collection of polynomial functions on Φ^m.

The ring of all polynomial functions from Φ^m to Φ will be denoted by the conventional symbol $\Phi[x_1, \ldots, x_m]$. Let

$$I(V) \subset \Phi[x_1, \ldots, x_m]$$

be the ideal consisting of those polynomials which vanish throughout V. The Hilbert "basis" theorem asserts that every ideal is spanned (as $\phi[x_1, \ldots, x_m]$-module) by some finite collection of polynomials. It follows that every algebraic set V can be defined by some finite collection of polynomial equations.

An important consequence of the Hilbert basis theorem is the following:

2.1 *Descending chain condition.* Any nested sequence $V_1 \supset V_2 \supset V_3 \supset \cdots$ of algebraic sets must terminate or stabilize $(V_i = V_{i+1} = V_{i+2} = \cdots)$ after a finite number of steps.

* It is customary in algebraic geometry to allow as "points" of V also m-tuples of elements belonging to some fixed algebraically closed extension field of Φ; but I do not want to allow this..

Note that the union $V \cup V'$ of any two algebraic sets V and V' in Φ^m is again an algebraic set.

A non-vacuous algebraic set V is called a *variety* or an *irreducible algebraic set* if it cannot be expressed as the union of two proper algebraic subsets. Note that V is irreducible if and only if $I(V)$ is a prime ideal. If V is irreducible, then the field of quotients f/g with f and g in the integral domain

$$\Phi[x_1, \ldots, x_m]/I(V)$$

is called the *field of rational functions* on V. Its transcendence degree over Φ is called the algebraic *dimension* of V over Φ.

If W is a proper subvariety of V, note that the dimension of W is less than the dimension of V. (See for example LANG, *Algebraic Geometry*, p. 29.)

Now let $V \subset \Phi^m$ be any non-vacuous algebraic set. Choose finitely many polynomials $f_1, \ldots, f_k$ which span the ideal $I(V)$ and, for each $x \in V$, consider the $k \times m$ matrix $(\partial f_i/\partial x_j)$ evaluated at x. Let ρ be the largest rank which this matrix attains at any point of V.

DEFINITION. A point $x \in V$ is called *non-singular* or *simple* if the matrix $(\partial f_i/\partial x_j)$ attains its maximal rank ρ at x; and *singular** if

$$\operatorname{rank}\,(\partial f_i(x)/\partial x_j) < \rho\ .$$

Note that this definition does not depend on the choice of $\{f_1, \ldots, f_k\}$. (For if we add an extra polynomial $f_{k+1} = g_1 f_1 + \cdots + g_k f_k$ the resulting new row in our matrix will be a linear combination of the old rows.)

LEMMA 2.2. *The set $\Sigma(V)$ of all singular points of V forms a proper algebraic subset (possibly vacuous) of V.*

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* This definition is certainly the correct one whenever V is a variety, or a union of varieties all of which have the same dimension. In other cases it does not correspond too well to intuitive expectations. For example if V is the union of a point and a line, then only the point is non-singular.

For a point x of V belongs to $\Sigma(V)$ if and only if every $\rho \times \rho$ minor determinant of $(\partial f_i/\partial x_j)$ vanishes at x. Thus $\Sigma(V)$ is determined by polynomial equations.

Now let us specialize to the case of a real or complex algebraic set.

THEOREM 2.3 (Whitney). *If Φ is the field of real (or complex) numbers, then the set $V-\Sigma(V)$ of non-singular points of V forms a smooth, non-vacuous manifold. In fact this manifold is real (or complex) analytic, and has dimension $m-\rho$ over Φ.*

The reader is referred to WHITNEY, *Elementary Structure of Real Algebraic Varieties,* for the elegant proof of 2.3.

In the case of an irreducible V, Whitney shows that *the dimension of the analytic manifold $V-\Sigma(V)$ over Φ is precisely equal to the algebraic dimension of V over Φ.*

Here is another basic result.

THEOREM 2.4 (Whitney). *For any pair $V \supset W$ of algebraic sets in a real or complex coordinate space, the difference $V-W$ has at most a finite number of topological components.*

For example, V itself has only finitely many components; and the smooth manifold $V-\Sigma(V)$ has only finitely many components.

A proof of 2.4, only slightly different from WHITNEY'S proof, will be given in Appendix A.

Here are three examples. (Compare Figure 3.) Each example will be a curve in the real plane having the origin as unique singular point.

EXAMPLE A. The variety consisting of all (x, y) in R^2 with

$$y^2 - x^2(1-x^2) = 0$$

illustrates the most well behaved and easily understood type of singular point, a "double point" at which two real analytic branches with distinct tangents (namely $y = x\sqrt{1-x^2}$ and $y = -x\sqrt{1-x^2}$) cross each other.*

* This can also be seen from the parametric representation $x = \sin\theta$, $2y = \sin 2\theta$ (which shows that the curve is a "Lissajous figure").

(Figure 3-A. For a definition of the term "branch" see §3.3.)

EXAMPLE B. The cubic curve

$$y^2 - x^2(x-1) = 0$$

of Figure 3-B has an isolated point at the origin; yet this curve is also irreducible.

(REMARK. Over the field of complex numbers, examples of this type cannot occur. In fact a theorem of RITT implies that the manifold of simple points of a complex variety V is everywhere dense in V. Compare VAN DER WAERDEN *Zur algebraische Geometrie III*, or *Algebraische Geometrie*, p. 134.)

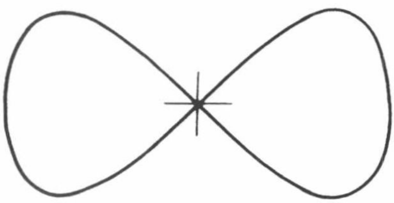

Figure 3-A. The curve $y = \pm x\sqrt{1-x^2}$

Figure 3-B. The curve $y = \pm x\sqrt{x-1}$

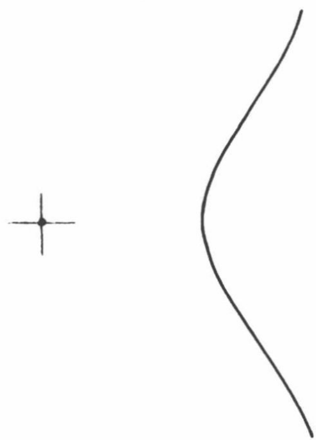

Figure 3-C. The curve $x^2 = y\left(1+\sqrt{1+y}\right)$

EXAMPLE C. The equation $y^3 = x^{100}$ can be solved for y as a 33-times differentiable function of x, yet this equation defines a variety $V \subset R^2$ which has a singular point at the origin. The equation $y^3 + 2x^2y - x^4 = 0$, which is illustrated in Figure 3-C, can actually be solved for y as a real